CATEGORIES

AND

DISTRIBUTIVELY GENERATED NEAR-RINGS

BY

SURAIYA JABEEN MAHMOOD

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Dedicated

to

the memory of my late father

Mohammad Bashir Qureshi

The following record of research work is submitted as a thesis for the degree of Doctor of Philosophy, in the University of Edinburgh, having been submitted for no other degree. Except where acknowledgement is made, the work is original.

Mrs. Suraiya Jabeen Mahmood

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Abstract

The notions of upper and lower faithful d.g. near-rings, first given by J.D.P. Meldrum [11], are generalized and used to prove that the category $\mathcal{B}$ of all faithful d.g. near-rings is a reflective as well as a coreflective subcategory of the category $\mathcal{A}$ of all d.g. near-rings. $\mathcal{A}$ and $\mathcal{B}$ are also shown to be complete and cocomplete.

Let $\mathcal{C}$ be an algebraic category satisfying certain conditions and let $\mathcal{D}$ be a reflective subcategory of $\mathcal{C}$ such that each reflection homomorphism is onto. In chapter 3 , it is proved that if this is the case, then the existence of limits in $\mathcal{D}$ implies the existence of limits in $\mathcal{C}$. An abstract version of this result is also given in this chapter.

If S is an inverse semigroup of endomorphisms of an additive group G , then S is shown to be an inverse semigroup of partial isomorphisms of G such that each $s \in S$ gives a splitting of G and these splittings are shown to satisfy certain conditions. This result and its converse are given in chapter 5 . Moreover, in this chapter, it is proved that for an inverse semigroup S, each d.g. near-ring (R,S) is faithful, having a faithful representation on its additive group $(R , +)$.

Throughout this work , a few categories are involved. In chapter 4, the existence of some natural functors, among these categories, is shown.

In the last chapter, the concept of a group d.g. near-ring, given by J.D.P. Meldrum [12] for a faithful d.g. near-ring , is generalized.

Introduction

A d.g. near-ring is not necessarily faithful. But J.D.P. Meldrum, in [11], proved that with each d.g. near-ring we can associate two faithful d.g. near-rings, namely the upper and lower faithful d.g. near-rings. In chapter 2, we generalize the notions of upper and lower faithful d.g. near-rings and make a correction in the latter case. Then using these ideas, we show that the category $\mathcal{B}$, of all faithful d.g. near-rings, is a reflective as well as a coreflective subcategory of the category $\mathcal{A}$ of all d.g. near-rings. Moreover we prove that the categories $\mathcal{A}$ and $\mathcal{B}$ are complete and cocomplete.

It may be interesting to mention that initially we proved the existence of products and general limits first in $\mathcal{B}$ and then, with the help of this result, we demonstrated the existence of products and general limits in $\mathcal{A}$, which is unusual. When we looked into the structure of the product object in $\mathcal{A}$, we noticed that the existence of products in $\mathcal{A}$ can be proved directly. But the original proof provided a general result of algebraic categories which satisfy certain conditions. We give this in chapter 3, where we also prove an abstract version of this result.

The generating semigroup plays an important role in the structure and behaviour of a d.g. near-ring; for example,

1) if S has a left identity e, then e is a left identity of (R,S) and so (R,S) is faithful having a faithful representation

on $(R, +)$,

2) if the generating semigroup S is a group, then (R,S) is faithful.

Therefore it was natural to think about the behaviour of d.g. near-rings generated by other types of semigroups, for example, inverse semigroups. J.D.P. Meldrum thought about this and conjectured that "every d.g. near-ring (R,S), over an inverse semigroup S, is faithful, having a faithful representation on $(R, +)$". We prove this conjecture in chapter 5. In that chapter, we also study the representations of an inverse semigroup and prove that, if an inverse semigroup S is contained in the semigroup of endomorphisms of a group G, then S is an inverse semigroup of partial isomorphisms of G such that each $s \in S$ gives a splitting of G, satisfying certain conditions. We prove the converse of this result as well.

In considering the behaviour of d.g. near-rings, we relate a few categories, namely the categories of all d.g. near-rings, of all faithful d.g. near-rings, of all semigroups, of all groups, of all sets and of (R,S)-groups for a d.g. near-ring (R,S). In chapter 4, we give a few natural functors among these categories, and their properties.

J.D.P. Meldrum, in [12], constructed a group d.g. near-ring $(R(G),SG)$ for a faithful d.g. near-ring (R,S) over a multiplicative group G. In chapter 6, we generalize this idea and construct a group d.g. near-ring for any d.g. near-ring (R,S) on a multiplicative group G, as a homomorphic image of $(\bar{R}(G), SG)$, where $(\bar{R}, S)$ is the upper faithful d.g. near-ring for (R,S).

For the contents of § 1.1, we refer to [13], except for where it is mentioned otherwise and the definition of an adjunction, for which the reference is made to [10]. Since we often work in the categories consisting of algebraic systems, we give some definitions and results in § 1.2, from [3], which deals with algebraic categories only.

Up to now there is only one book published on near-rings [15], in which almost all the material available on this subject is gathered systematically. So we refer to this book in general for the contents of §§ 1.4., 1.5, and 1.6. But occasionally we mention the alternative sources, where the topics we need are not covered fully in this book. Note that a few of our terms are different from that of [15].

In [10], [13] and [15] the maps are written on the left. But, to be consistent with our work to follow, we have shifted them to the right.

For the contents of § 5.1 we refer to [9], [1] and [2], and for that of § 6.1 we refer to [12]. For the results of group theory, which we have used, reference is made to [16] and [14].

Contents

Chapter 1

Preliminaries

1.1. Categories and Functors

Definition 1.1.1. A category $\mathcal{A}$ is a class of objects together with a class $\bigcup_{(A,B) \in \mathcal{A} \times \mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(A,B)$ which is a disjoint union of sets $\mathrm{Hom}_{\mathcal{A}}(A,B)$, possibly empty, of arrows, called morphisms with domain A and codomain B , for all pairs A , B of objects of $\mathcal{A}$. Furthermore for objects A , B , C of $\mathcal{A}$ and $\alpha \in \mathrm{Hom}_{\mathcal{A}}(A,B)$, $\beta \in \mathrm{Hom}_{\mathcal{A}}(B,C)$ a composition is defined as $\alpha\beta \in \mathrm{Hom}_{\mathcal{A}}(A,C)$ which satisfies :

1) whenever the compositions make sense we have

$$(\alpha\beta)\gamma = \alpha(\beta\gamma)$$

2) for each object $A \in \mathcal{A}$, there exists $I_A \in \mathrm{Hom}\,(A,A)$ such that $\alpha I_A = \alpha$, $I_A\beta = \beta$ whenever the compositions make sense.

A category $\mathcal{A}$ is called a small category if the class of objects of $\mathcal{A}$ is a set. In this case the class of morphisms $\bigcup_{(A,B) \in \mathcal{A} \times \mathcal{A}} \mathrm{Hom}_{\mathcal{A}}(A,B)$, being a union of sets indexed over a set , is also a set.

We normally write $A \in \mathcal{A}$ to mean that A is an object in $\mathcal{A}$ and $\alpha : A \longrightarrow B$ to mean that $\alpha \in \mathrm{Hom}_{\mathcal{A}}(A,B)$.

Examples 1.1.2.

1) The collection $\mathcal{G}$ of all groups with group homomorphisms.

2) The collection $\underline{A}$ of all abelian groups with group homomorphisms.

3) The collection of all sets and set mappings.

4) The collection of all near-rings and near-ring homomorphisms.

5) The collection of all d.g. near-rings and d.g. near-ring homomorphisms.

Definition 1.1.3. A category $\mathcal{B}$ is called a subcategory of a category $\mathcal{A}$ if

1) $\mathcal{B} \subseteq \mathcal{A}$, i.e. , the objects of $\mathcal{B}$ are also objects of $\mathcal{A}$.

2) $\mathrm{Hom}_{\mathcal{B}}(A,B) \subseteq \mathrm{Hom}_{\mathcal{A}}(A,B)$, for all objects A , B in $\mathcal{B}$.

3) The composition of two morphisms in $\mathcal{B}$ is the same as their composition in $\mathcal{A}$.

4) $I_A \in \mathrm{Hom}_{\mathcal{B}}(A,A)$ is the same in $\mathcal{B}$ as in $\mathcal{A}$ for all $A \in \mathcal{B}$.

If furthermore $\mathrm{Hom}_{\mathcal{B}}(A,B) = \mathrm{Hom}_{\mathcal{A}}(A,B)$ for all A , $B \in \mathcal{B}$, then $\mathcal{B}$ is called a full subcategory of $\mathcal{A}$.

Example 1.1.2.(2) is a full subcategory of 1.1.2.(1) whereas 1.1.2.(5) is a subcategory of 1.1.2.(4) which is not full.

Definition 1.1.4. The dual category $\mathcal{A}^*$ of a category $\mathcal{A}$ has the same class of objects as $\mathcal{A}$ and is such that

$$\mathrm{Hom}_{\mathcal{A}}(A,B) = \mathrm{Hom}_{\mathcal{A}^*}(B,A)$$

for all A , B in $\mathcal{A}$. The composition $\beta\alpha$ in $\mathcal{A}^*$ is defined as the composition $\alpha\beta$ in $\mathcal{A}$. Clearly $(\mathcal{A}^*)^* = \mathcal{A}$ and consequently every result about categories actually embodies two results. If a statement p is true for a category $\mathcal{A}$, then there is a dual statement p^* which will be true for $\mathcal{A}^*$. If the assumptions on $\mathcal{A}$ used to prove p hold also in $\mathcal{A}^*$, then p^* is true for $(\mathcal{A}^*)^* = \mathcal{A}$.

Definition 1.1.5. A morphism $\alpha \in \text{Hom}(A,B)$ is called a monomorphism if $f\alpha = g\alpha$ implies that $f = g$ for all morphisms f, g with codomain A. α is called an epimorphism if $\alpha f = \alpha g$ implies that $f = g$ for all morphisms f, g with domain B. In other words α is a monomorphism (epimorphism) if α is right(left) cancellable.

The notion of an epimorphism is dual to that of a monomorphism.

Theorem 1.1.6. a) If α, β are epimorphisms (monomorphisms) and $\alpha\beta$ is defined then $\alpha\beta$ is an epimorphism (a monomorphism).
b) If $\alpha\beta$ is an epimorphism (a monomorphism) then β (α) is an epimorphism (a monomorphism).

Definition 1.1.7. A morphism $\theta : A \longrightarrow B$ is said to be an isomorphism if there exists a morphism $\theta' : B \longrightarrow A$ such that $\theta\theta' = I_A$ and $\theta'\theta = I_B$. A morphism (isomorphism) $\alpha : A \longrightarrow A$ is called an endomorphism (automorphism) of A.

Definition 1.1.8. In a category, if $\alpha : A' \longrightarrow A$ is a monomorphism, A' is called a subobject of A and we refer to α as the inclusion of A' in A. Sometimes we write $\alpha : A' \subseteq A$. If α is not an isomorphism we call A', a proper subobject of A. The composition of a monomorphism $\alpha : A' \longrightarrow A$ with a morphism $f : A \longrightarrow B$ is often denoted by f / A' and is called the restriction of f to A'.

Definition 1.1.9. Given two morphisms $\alpha, \beta : A \longrightarrow B$ in a category $\mathcal{A}$, a morphism $\gamma : K \longrightarrow A$ is called an equalizer of α and β if $\gamma\alpha = \gamma\beta$ and if for every morphism $\gamma' : K' \longrightarrow A$ with $\gamma'\alpha = \gamma'\beta$ there exists a unique morphism $\mu : K' \longrightarrow K$ such that $\mu\gamma = \gamma'$. We write $K = \mathrm{Equ}(\alpha,\beta)$.

Usually K is called the equalizer and γ is viewed as an inclusion map because of the following result :

Theorem 1.1.10 If γ is an equalizer of α, β then γ is a monomorphism. Any two equalizers of α, β are isomorphic subobjects of A.

Definition 1.1.11. Let $\{ u_\lambda : A_\lambda \longrightarrow A \}_{\lambda \in \Lambda}$ be a family of subobjects of A. We call a morphism $u : A' \longrightarrow A$ the intersection of the family if for each $\lambda \in \Lambda$ there exists a unique morphism $\nu_\lambda : A' \longrightarrow A_\lambda$ such that $u = \nu_\lambda u_\lambda$ and furthermore if every morphism $\nu : B \longrightarrow A$ factoring through each u_λ, factors uniquely through u.

Consider a diagram

$$\begin{array}{ccc} A' & & B' \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & B \end{array}$$

where f is any morphism and the vertical morphisms are monomorphisms. Then we say that the subobject A' is carried into the subobject B' by f if there exists a morphism $g : A' \longrightarrow B'$ (necessarily unique) making the above diagram commutative.

Definition 1.1.12. The union of a family $\{ A_\lambda : \lambda \in \Lambda \}$ of subobjects of an object A is defined as a subobject A' of A which is preceded by each of the A_λ, and which has the following property: If $f : A \longrightarrow B$ is a morphism and each A_λ is carried into some subobject B' by f, then A' is also carried into B' by f. The object A' is denoted by $\cup \{ A_\lambda : \lambda \in \Lambda \}$.

Definition 1.1.13. Let $f : A \longrightarrow B$ be in $\mathcal{A}$. Then the image of f is defined as the smallest subobject of B which f factors through; that is a monomorphism $u : I \longrightarrow B$ is the image of f if $f = f'u$ for some $f' : A \longrightarrow I$ and if u precedes any other monomorphism into B with the same property. The object I is usually denoted by $\operatorname{Im} f$.

Definition 1.1.14. An object $0 \in \mathcal{A}$ is called a zero object if $\operatorname{Hom}(0,A)$ and $\operatorname{Hom}(A,0)$ have precisely one morphism each, for each $A \in \mathcal{A}$. A morphism $f : A \longrightarrow B$ in $\mathcal{A}$ is a zero morphism if it factors through the zero object.

For the following two definitions we assume that the category has a zero object.

Definition 1.1.15. [10] The cokernel of $f : A \longrightarrow B$ is a morphism $g : B \longrightarrow D$ such that

1) $f\,g = 0 : A \longrightarrow D$,

2) if there is a morphism $h : B \longrightarrow C$ such that $f\,h = 0$, then $h = g\,h'$ for a unique morphism $h' : D \longrightarrow C$.

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[dr, "h"'] & D \arrow[d, dashed, "h'"] \\ & & C \end{tikzcd} \qquad \begin{array}{l} f\,g = 0 \\ \\ f\,h = 0 \end{array}$$

Cokernels are epimorphisms.

Definition 1.1.16. For $f : A \longrightarrow B$ in $\mathcal{A}$, we call a morphism $u : K \longrightarrow A$, the kernel of f , if $u\,f = 0$ and if for every morphism $u' : K' \longrightarrow A$ with $u'f = 0$, there exists a unique morphism $\gamma : K \longrightarrow K'$ such that $\gamma\,u = u'$.

$$\begin{tikzcd} K \arrow[r, "u"] & A \arrow[r, "f"] & B \\ K' \arrow[u, dashed, "\gamma"] \arrow[ur, "u'"'] & & \end{tikzcd} \qquad \begin{array}{l} u\,f = 0 \\ \\ u'f = 0 \end{array}$$

Kernels are monomorphisms. We usually write $K = \operatorname{Ker} f$.

Definition 1.1.17. Let $\{ A_\lambda : \lambda \in \Lambda \}$ be a family of objects in a category $\mathcal{A}$. An object $A \in \mathcal{A}$ together with a family $\{ p_\lambda : A \longrightarrow A_\lambda \}_{\lambda \in \Lambda}$ of morphisms is the product of $\{ A_\lambda : \lambda \in \Lambda \}$ if whenever there is a family

$\{ \alpha_\lambda : B \longrightarrow A_\lambda \}_{\lambda \in \Lambda}$ of morphisms in $\mathcal{A}$, then there exists a unique morphism $\phi : B \longrightarrow A$ such that $\phi\, p_\lambda = \alpha_\lambda$ for each $\lambda \in \Lambda$. We write $A = \prod_{\lambda \in \Lambda} A_\lambda$ and call the p_λ's projections.

Dually we define the coproduct $\{ u_\lambda : A_\lambda \longrightarrow A \}_{\lambda \in \Lambda}$ in $\mathcal{A}$ of $\{ A_\lambda : \lambda \in \Lambda \}$. We write $A = \coprod_{\lambda \in \Lambda} A_\lambda$ or $\ast_{\lambda \in \Lambda} A_\lambda$ and call the u_λ's injections.

Definition 1.1.18 A triple (Λ, M, d), where Λ is a set whose elements are called vertices, M a set whose elements are called arrows and d a function $: M \longrightarrow \Lambda \times \Lambda$, is called a diagram scheme. If $m\, d = (\lambda, \mu)$ we say λ is the origin and μ the extremity of m.

Definition 1.1.19 Let $\mathcal{A}$ be a category. A diagram D in $\mathcal{A}$ over a scheme (Λ, M, d) is a function which assigns to each vertex $\lambda \in \Lambda$ an object A_λ of $\mathcal{A}$ and to each arrow m with origin λ and extremity μ a morphism $m\, D = m_{\lambda\mu} \in \mathrm{Hom}_{\mathcal{A}}(A_\lambda, A_\mu)$.

Definition 1.1.20 A diagram, in which any two different finite compositions of morphisms with same origin and extremity are equal, is called a commutative diagram.

Definition 1.1.21 If D is a diagram in $\mathcal{A}$ over a scheme (Λ, M, d), we say that a family $\{ \alpha_\lambda : A \longrightarrow A_\lambda \}_{\lambda \in \Lambda}$ of morphisms is compatible for D if for every arrow $m \in M$ the diagram

$$\begin{tikzcd} & A_\lambda \arrow[dd, "m_{\lambda\mu}"] \\ A \arrow[ur, "\alpha_\lambda"] \arrow[dr, "\alpha_\mu"'] & \\ & A_\mu \end{tikzcd}$$

is commutative.

Definition 1.1.22 A family $\{ \alpha_\lambda : A \longrightarrow A_\lambda \}_{\lambda \in \Lambda}$ is a limit for D in $\mathcal{A}$ if it is compatible for D and if for every compatible family $\{ \beta_\lambda : B \longrightarrow A_\lambda \}_{\lambda \in \Lambda}$ for D in $\mathcal{A}$, there exists a unique morphism $\phi : B \longrightarrow A$ such that $\phi\, \alpha_\lambda = \beta_\lambda$ for each $\lambda \in \Lambda$. If $\{ \beta_\lambda : B \longrightarrow A_\lambda \}_{\lambda \in \Lambda}$ is also a limit for D then ϕ is an isomorphism.

If a category $\mathcal{A}$ has all limits for all diagram schemes, then it is called a complete category.

Dually we can define the notions of cocompatible family, colimit for a diagram and cocomplete category.

Definition 1.1.23 Let $\mathcal{A}$ and $\mathcal{B}$ be two categories. A covariant functor $T : \mathcal{A} \longrightarrow \mathcal{B}$ is a function which assigns to each object $A \in \mathcal{A}$ an object $A\,T \in \mathcal{B}$, and to each morphism $\alpha : A \longrightarrow B$ in $\mathcal{A}$ a morphism $\alpha\, T : A\,T \longrightarrow B\,T$ such that

1) if $\alpha\beta$ is defined in $\mathcal{A}$, then $(\alpha\beta)T = (\alpha T)(\beta T)$,

2) $I_A T = I_{A T}$ for each $A \in \mathcal{A}$.

Definition 1.1.24 If $\mathcal{A}'$ is a subcategory of $\mathcal{A}$, then the covariant functor $I : \mathcal{A}' \longrightarrow \mathcal{A}$ such that $A I = A$ for all $A \in \mathcal{A}'$, and $\alpha I = \alpha$ for all morphisms α in $\mathcal{A}'$ is called the inclusion functor of $\mathcal{A}'$ in $\mathcal{A}$.

Definition 1.1.25 Let S , T be two covariant functors from a category $\mathcal{A}$ to a category $\mathcal{B}$. Suppose that for each $A \in \mathcal{A}$, there is a morphism $\eta_A : A S \longrightarrow A T$ such that for every morphism $\alpha : A \longrightarrow B$ in $\mathcal{A}$ the diagram

$$\begin{tikzcd}
A S \arrow[r, "\eta_A"] \arrow[d, "\alpha S"'] & A T \arrow[d, "\alpha T"] \\
B S \arrow[r, "\eta_B"] & B T
\end{tikzcd}$$

is commutative. Then $\eta : S \longrightarrow T$ is called a natural transformation from S to T. If each η_A is an isomorphism then η is called a natural equivalence.

Definition 1.1.26 Let $\mathcal{A}$ and $\mathcal{B}$ be two categories together with two covariant functors $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{A}$. Suppose that there is a function ϕ which assigns to each pair (A,B)

of $\mathcal{A} \times \mathcal{B}$ a bijection $\phi_{A,B}$ which in turn assigns to each morphism $f : A \longrightarrow B\,G$ a morphism $f\,\phi_{A,B} = f\,\phi : A\,F \longrightarrow B$ such that for all morphisms $h : A' \longrightarrow A$, $k : B \longrightarrow B'$,

$$(h\,f)\phi = h\,F \,.\, f\phi \quad , \quad (f \,.\, k\,G)\phi = f\phi \,.\, k \ .$$

Then we say that (F,G,ϕ) is an adjunction and F is the coadjoint of G and G is the adjoint of F .

Remark 1.1.27 We are using the terms of adjoint and coadjoint in the sense of Mitchell [13].

Definition 1.1.28 Let $\mathcal{A}'$ be a subcategory of a category $\mathcal{A}$. A coreflection for $A \in \mathcal{A}$ in $\mathcal{A}'$ is an object $A\,T \in \mathcal{A}'$ together with a morphism

$$\theta_A : A \longrightarrow A\,T$$

such that for every object $A' \in \mathcal{A}'$ and every morphism $\alpha : A \longrightarrow A'$ there exists a unique morphism $\beta : A\,T \longrightarrow A'$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "\theta_A"] \arrow[dr, "\alpha"'] & A\,T \arrow[d, dashed, "\beta"] \\ & A' \end{tikzcd}$$

is commutative . Equivalently , denoting by I the inclusion functor : $\mathcal{A}' \longrightarrow \mathcal{A}$, $\theta_A : A \longrightarrow A\,T$ defines a coreflection of A in $\mathcal{A}'$ if the function

$$\operatorname{Hom}_{\mathcal{A}'}(AT\,,\,A') \longrightarrow \operatorname{Hom}_{\mathcal{A}}(A\,,\,A'I)$$

induced by θ_A is a one to one correspondence for all $A' \in \mathcal{A}'$. From the uniqueness of the morphism $\beta : AT \longrightarrow A'$ it follows that any two coreflections of A in $\mathcal{A}'$ are isomórphic. If $\mathcal{A}'$ is a full subcategory of $\mathcal{A}$, then every object of $\mathcal{A}$ which is in $\mathcal{A}'$ is its own coreflection via the identity morphism.

Dually, for $A \in \mathcal{A}$, $\theta_A : AT \longrightarrow A$ is called the reflection of A in $\mathcal{A}'$ if $AT \in \mathcal{A}'$ and if for every morphism $\alpha : A' \longrightarrow A$ with $A' \in \mathcal{A}'$ there is a unique morphism $\beta : A' \longrightarrow AT$ making the diagram

$$\begin{tikzcd}
AT \arrow[r, "\theta_A"] & A \\
 & A' \arrow[u, "\alpha"'] \arrow[ul, dashed, "\beta"]
\end{tikzcd}$$

commutative.

If every object of $\mathcal{A}$ has a coreflection (reflection) in $\mathcal{A}'$, then $\mathcal{A}'$ is called a coreflective (reflective) subcategory of $\mathcal{A}$. In this case $T : \mathcal{A} \longrightarrow \mathcal{A}'$ becomes a covariant functor, called the coreflector (reflector) of $\mathcal{A}$ in $\mathcal{A}'$, which assigns to each morphism $\alpha : A \longrightarrow B$ of $\mathcal{A}$ the unique morphism

$\alpha T : AT \longrightarrow BT$ in $\mathcal{A}'$ such that the diagram

$$\begin{tikzcd} A \arrow[r, "\theta_A"] \arrow[d, "\alpha"'] & AT \arrow[d, "\alpha T"] \\ B \arrow[r, "\theta_B"] & BT \end{tikzcd} \qquad \begin{tikzcd} A \arrow[d, "\alpha"'] & AT \arrow[l, "\theta_A"'] \arrow[d, "\alpha T"] \\ B & BT \arrow[l, "\theta_B"] \end{tikzcd}$$

is commutative. It is easy to check the naturality of the 1 - 1 correspondence $\phi : \mathrm{Hom}_{\mathcal{A}'}(AT , A') \longrightarrow \mathrm{Hom}_{\mathcal{A}}(A , A'I)$ ($\phi : \mathrm{Hom}\ (A' , AT) \longrightarrow \mathrm{Hom}\ (A'I , A)$) in both A and A'. Therefore we have :

Theorem 1.1.29 If $T : \mathcal{A} \longrightarrow \mathcal{A}'$ is a coreflector (reflector) functor then T is a coadjoint (adjoint) of the inclusion functor $I : \mathcal{A}' \longrightarrow \mathcal{A}$.

Conversely, if the inclusion functor $I : \mathcal{A}' \longrightarrow \mathcal{A}$ has a coadjoint (adjoint) T then T is the coreflector (reflector) of $\mathcal{A}$ in $\mathcal{A}'$.

Theorem 1.1.30 Let $\mathcal{A}'$ be a full reflective subcategory of a category $\mathcal{A}$. If a diagram D in $\mathcal{A}'$ over a scheme (Λ, M, d), has a colimit $\{ \alpha_\lambda : A_\lambda \longrightarrow A \}_{\lambda \in \Lambda}$ in $\mathcal{A}$, then $AT \cong A$ and $\{ \alpha_\lambda : A_\lambda \longrightarrow AT \}_{\lambda \in \Lambda}$ is a colimit of D in $\mathcal{A}'$, T being the reflector of $\mathcal{A}$ in $\mathcal{A}'$.

Theorem 1.1.31 Let $\mathcal{A}'$ be a full reflective subcategory of a category $\mathcal{A}$. If a diagram D in $\mathcal{A}'$ over a scheme (Λ , M, d), has a limit $\{ p_\lambda : A \longrightarrow A_\lambda \}_{\lambda \in \Lambda}$ in $\mathcal{A}$ then it has a limit $\{ \theta p_\lambda : AT \longrightarrow A_\lambda \}_{\lambda \in \Lambda}$ in $\mathcal{A}'$, where $\theta : AT \longrightarrow A$ is the reflection of A in $\mathcal{A}'$, with T the reflector of $\mathcal{A}$ in $\mathcal{A}'$.

Definition 1.1.32. Let a category $\mathcal{A}$ have a property p, where p can be intersections (unions, images, kernels, cokernels, intersection of kernels) . Then we say that $\mathcal{A}$ is a category with p.

1.2. Algebraic Categories

By an algebraic category we mean a category with objects having certain algebraic structure or structures, for example, rings, groups, modules, near-rings etc. , and having as morphisms the structure preserving mappings. We call morphisms in this case, homomorphisms (of the structure of the objects). We say a homomorphism is an epimorphism (monomorphism, isomorphism) if it is a surjection (injection, bijection). We note that an epimorphism (monomorphism, isomorphism) is also an epimorphism (monomorphism, isomorphism) in the sense of 1.1.5.

Throughout this section $\mathcal{A}$ denotes an algebraic category.

The notions of binary relation and an equivalence relation are defined as usual.

Definition 1.2.1. An equivalence relation ρ on $A \in \mathcal{A}$ is called a congruence if it is a subobject of $A \times A$.

Definition 1.2.2. The composition of two binary relations ϕ and ψ on $A \in \mathcal{A}$ is defined as follows :

$$\phi \circ \psi = \{ (x,y) \in A \times A \ : \ (x,z) \in \phi \ , \ (z,y) \in \psi \ \text{ for some } z \in A \}.$$

Let $f : A \longrightarrow B$ be a homomorphism in $\mathcal{A}$. Then $\sigma = f \circ f^{-1}$ is an equivalence relation on A.

Definition 1.2.3. σ in the above result is called the kernel of f and is written as $\text{Ker } f$.

Theorem 1.2.4. The kernel of a homomorphism is a congruence.

For $A \in \mathcal{A}$ and for any congruence σ on A, A / σ is an object of $\mathcal{A}$ and $\sigma^{\natural} : A \longrightarrow A / \sigma$ is an epimorphism with $\text{Ker } \sigma^{\natural} = \sigma$.

Theorem 1.2.5. Let $f : A \longrightarrow B$ be an epimorphism and $g : A \longrightarrow C$ be any homomorphism such that $\text{Ker } f \subseteq \text{Ker } g$. Then there exists a unique homomorphism $h : B \longrightarrow C$ such that $f h = g$.

1.3. The Category $\mathcal{G}$ of all Groups

Let $G \in \mathcal{G}$. If G is generated as a group by a subset X , we write $G = Gp\{X\}$. In this case each element g of G is written as $g = \epsilon_1 x_1 + \epsilon_2 x_2 + ------ + \epsilon_n x_n$, $x_i \in X$, $\epsilon_i = \pm 1$, $i = 1, ------ n$, n a finite integer.

Definition 1.3.1. An element $\epsilon_1 x_1 + ------ + \epsilon_n x_n$ of $G = Gp\{X\}$ is called a relation in G relative to X if it is 0 in G .

Definition 1.3.2. Let $G = Gp\{X\}$ and H be any group . Then we call a mapping $\alpha : X \longrightarrow H$ well-defined if whenever $\epsilon_1 x_1 + ------ + \epsilon_n x_n$ is a relation in G relative to X , $\epsilon_1 x_1 \alpha + ------ + \epsilon_n x_n \alpha$ is 0 in H .

Theorem 1.3.3. Let $G = Gp\{X\}$ and H be any group . Then a well-defined mapping $\alpha : X \longrightarrow H$ can be extended to a group homomorphism $\bar{\alpha} : G \longrightarrow H$.

The category $\mathcal{G}$ is complete and cocomplete. In this case the coproducts are called free products and we write $G = \underset{\lambda \in \Lambda}{*} H_\lambda$ to mean that G is the free product of the family $\{ H_\lambda : \lambda \in \Lambda \}$ of groups.

Theorem 1.3.4. Let G be the free product of a family $\{ H_\lambda : \lambda \in \Lambda \}$ of groups. Then each non-zero element g of G is uniquely expressible as

$$g = h_{\lambda_1} + \cdots\cdots + h_{\lambda_r} ,$$

$$0 \neq h_{\lambda_i} \in H_{\lambda_i} , \quad \lambda_i \neq \lambda_{i+1} \quad i = 1 , \cdots , r-1 .$$

Theorem 1.3.5. Let $G = H * K$, for G , H and $K \in \mathcal{G}$, and let N be a normal subgroup of K . Then

$$G / N^G \cong H * (K / N) ,$$

where N^G is the normal closure of N in G .

Note. The results of § 1.2 and § 1.3 will be used frequently and since they are well known we will not mention them explicitly .

1.4. Near-Rings

Definition 1.4.1. A (left) near-ring is a set R together with two binary operations $+$ and $\cdot$ such that

1) $(R,+)$ is a group , not necessarily commutative ,

2) $(R,\cdot)$ is a semigroup ,

3) for all r_1 , r_2 and r_3 in R ,

$$r_1(r_2 + r_3) = r_1 r_2 + r_1 r_3 ,$$

i.e. , the elements of R satisfy the left distributive law.

Examples 1.4.2.

1) The set $T(G)$ of all mappings of a group G into itself with pointwise addition and multiplication as composition of mappings ,

forms a near-ring.

2) The subset $T_0(G)$ of $T(G)$ consisting of all mappings which map 0 onto 0, is also a near-ring.

3) Let R be a commutative ring with identity. Then the set $R[x]$ of polynomials with usual addition and with the operation of substitution as multiplication, forms a near-ring.

4) Let G be a non-abelian group and $End(G)$ be the semigroup of endomorphisms of G. $End(G)$ is not closed under addition, but the set of all finite sums and differences of the elements of $End(G)$ is closed under addition and multiplication, and forms a near-ring. We denote this near-ring by $E(G)$.

Theorem 1.4.3. In a near-ring R,

$$r\,0 = 0 \quad \text{and} \quad r_1(-r) = -r_1 r,$$

for all $r_1, r \in R$.

Remark 1.4.4. In general $0\,r \neq 0$ and $(-r)\,r' \neq -r\,r'$. For example in $T(G)$, $0 \cdot f \neq 0$ and $(-f) \cdot f' \neq -(f \cdot f')$ in general.

Definition 1.4.5. A near-ring R is said to be zero symmetric if $0\,r = 0$ for all $r \in R$.

$T_0(G)$ in 1.4.2.(2) is a zero symmetric near-ring.

Definition 1.4. 6. A mapping α from a near-ring R to a near-ring T is called a near-ring homomorphism if

1) α is a group homomorphism from $(R,+)$ to $(T,+)$,

2) α is a semigroup homomorphism from $(R,.)$ to $(T,.)$.

We can define epimorphism , monomorphism and isomorphism in the usual way.

Definition 1.4.7. A subset S of R is said to be a sub-near-ring of the near-ring R if S itself is a near-ring with respect to the same addition and multiplication.

$T_0(G)$ is a sub near-ring of $T(G)$.

Definition 1.4.8. We call a normal subgroup I of a near-ring R an ideal if

1) $R I \subseteq I$

2) for all $r , r' \in R$ and all $a \in I$

$$(r + a) r' - r r' \in I$$

A normal subgroup of a near-ring R satisfying (1) is called a left ideal of R , while a normal subgroup satisfying (2) is called a right ideal of R . R is an ideal of R .

It is easy to see that I is an ideal of a near-ring R if and only if $r \equiv r' (\mathrm{mod}\ I)$ is a congruence relation on R . For every ideal I , the quotient set R / I is a near-ring.

Theorem 1.4.9. (Homomorphism Theorem)

1) If I is an ideal of a near-ring R then the canonical map

$\pi : R \longrightarrow R / I$ is a near-ring homomorphism and R / I is a homomorphic image of R.

2) Conversely, if $\alpha : R \longrightarrow T$ is a near-ring epimorphism then $R / \mathrm{Ker}\, \alpha \cong T$.

Theorem 1.4.10 Let $\alpha : R \longrightarrow T$ be a near-ring epimorphism. Then there is a one to one, order preserving, correspondence between the sub-near-rings (ideals) of R containing $\mathrm{Ker}\, \alpha$ and the sub-near-rings (ideals) of T.

If $\pi : R \longrightarrow R / I$ is the canonical epimorphism then for all ideals J of R, such that $J \supset I$,

$$\pi(R) / \pi(J) \cong R / J .$$

Theorem 1.4.11. If I and J are ideals of a near-ring R, then $I \cap J$ is an ideal of R and $J + I / I \cong J / I \cap J$.

Definition 1.4.12. A group G is called an R-group, R a near-ring, if there exists a mapping $\mu : G \times R \longrightarrow G$ defined by $(g, r)\mu = gr$ such that for all $r, r' \in R$, $g \in G$

$$g(r + r') = gr + gr'$$

$$g(r r') = (gr)r' .$$

Obviously $(R,+)$ is an R-group for every near-ring R.

1.5. D.G. Near-Rings

Definition 1.5.1. An element d of a near-ring R is said to be distributive if $(r + r')d = rd + r'd$ for all $r, r' \in R$. A near-ring in which every element is distributive is called a distributive near-ring.

The endomorphisms of G are distributive in the near-ring $T_0(G)$ (1.4.2.(2)).

Remark 1.5.2. In a near-ring the set of distributive elements forms a multiplicative semigroup.

Definition 1.5.3. A near-ring R is said to be a d.g. near-ring (distributively generated near-ring) if $(R,+)$ is generated by a semigroup S of distributive elements of R. S need not be the whole set of distributive elements of R. Because of the importance of the generating semigroup we write a d.g. near-ring as (R,S).

For any non abelian group G, $(E(G),End(G))$ is a d.g. near-ring.

Remark 1.5.4. Every element r of a d.g. near-ring (R,S) is a finite sum of the form $r = \epsilon_1 s_1 + - - - - - - - + \epsilon_n s_n$, $\epsilon_i = \pm 1$, $s_i \in S$, $i = 1, - - - - -, n$.

We will represent $r \in (R,S)$ in this way without further reference.

Theorem 1.5.5. Let (R,S) be a d.g. near-ring. Then

1) $\forall$ $r \in (R,S)$ and $\forall$ $s \in S$

$$(-r)s = -(rs) \quad , \quad s(-r) = -(sr)$$

2) (R,S) is zero symmetric ,

3) $\forall\ r , r' \in (R,S)$ and $\forall\ s \in S$

$$(r + r')(-s) = -(r + r')s = -(rs + r's) = -r's - rs ,$$

4) if $r = \sum_{i=1}^{n} \epsilon_i s_i$, $r' = \sum_{j=1}^{m} \mu_j t_j$, $s_i , t_j \in S$

$$r r' = \sum_{j=1}^{m} \mu_j \left(\sum_{i=1}^{n} \epsilon_i s_i t_j \right) .$$

Theorem 1.5.6. Let (R,S) be a d.g. near-ring . Then

1) R is distributive if and only if $\forall\ a , b , c , d \in R$,

$$a b + c d = c d + a b ,$$

i.e. , if and only if R^2 is abelian ,

2) $(R,+)$ is abelian if and only if $(R,+,.)$ is a ring.

Theorem 1.5.7. Let (R,S) be a d.g. near-ring. Then

1) if S contains a left (right, two-sided) identity e then e is a left (right, two sided) identity for R .

2) if R contains exactly one left (right) identity then it is the two-sided identity of R .

Theorem 1.5.8. 1) If a near-ring R contains a d.g. near-ring T then R is not necessarily d.g.

2) A sub-near-ring of a d.g. near-ring need not be d.g.

3) Every homomorphic image of a d.g. near-ring is a d.g. near-ring.

Theorem 1.5.9. Let (R,S) be a d.g. near-ring. A normal subgroup I of $(R,+)$ is an ideal of (R,S) if and only if

1) $R\,I \subseteq I$,

2) $I\,S \subseteq I$.

Theorem 1.5.10 [11] Let (R,S) be a d.g. near-ring , X a subset of (R,S) . The ideal generated by X is the normal subgroup of $(R,+)$ generated by

$$R\,X\,S = \{ rxs , rx , xs , x : r \in R , x \in X , s \in S \}.$$

Definition 1.5.11. Let (R,S) and (T,U) be two d.g. near-rings. A mapping $\alpha : (R,S) \longrightarrow (T,U)$ is called a d.g. near-ring homomorphism if α is a near-ring homomorphism from R to T which maps S into U .

Remark 1.5.12. The kernel of a d.g. near-ring homomorphism is an ideal and one can state the homomorphism and isomorphism theorems for d.g. near-rings.

Theorem 1.5.13. [11] Let (R,S) and (T,U) be two d.g. near-rings. A group homomorphism $\alpha : (R,+) \longrightarrow (T,+)$ is a d.g. near-ring homomorphism from (R,S) to (T,U) if α is a semigroup homomorphism from S to U .

Let S be a multiplicative semigroup . Let $(Fr(S),+)$ be the free group on the set S . This group consists of all finite sums $\sum_i \epsilon_i s_i$. By defining

$$\left(\sum_i \epsilon_i s_i \right) \cdot \left(\sum_j \mu_j t_j \right) = \sum_j \mu_j \left(\sum_i \epsilon_i s_i t_j \right)$$

we get :

Theorem 1.5.14. 1) The multiplication $\cdot$ is well defined.

2) $(Fr(S),+,\cdot)$ is a d.g. near-ring generated by S .

3) For every d.g. near-ring (T,U) , every semigroup homomorphism $\theta : S \longrightarrow U$ can be extended uniquely to a d.g. near-ring homomorphism : $(Fr(S),S) \longrightarrow (T,U)$.

4) Every d.g. near-ring (R,S) is a homomorphic image of $(Fr(S),S)$.

5) $(Fr(S),S)$ is determined uniquely to within d.g. near-ring isomorphism.

Definition 1.5.15. $(Fr(S),S)$ is called the free d.g. near-ring on the multiplicative semigroup S .

1.6. Representations

Definition 1.6.1.(1) Let (R,S) be a d.g. near-ring . A group G is called an (R,S)-group if there exists a d.g. near-ring homomorphism $\theta : (R,S) \longrightarrow (E(G),End(G))$. The map θ is often omitted and we write gr for $g(r\theta)$, $g \in G$, $r \in (R,S)$.

2) Let S be a multiplicative semigroup. A group G is said to be an S-group if there exists a semigroup homomorphism θ from S to $End(G)$ and we write gs for the element $g(s\theta)$ of G .

In (1) $\{$ (2) $\}$ θ is said to be a representation of (R,S)

$\{ S \}$ on G.

Remark 1.6.2. If (R,S) is a d.g. near-ring, then an R-group G is an (R,S)-group if and only if G is an S-group under the induced representation.

Theorem 1.6.3. [6] Let (R,S) be a d.g. near-ring and G an (R,S)-group. A subgroup H of G is an (R,S)-group if and only if it is an S-group.

Theorem 1.6.4. [7] 1) Every S-group G, S a multiplicative semigroup, is an $(Fr(S),S)$ group.

2) Let $(R,S) \cong (Fr(S),S) / I$ be a d.g. near-ring. Then an S group G is an (R,S)-group if and only if $G I = \{0\}$.

Definition 1.6.5. Let H and G be (R,S)-groups $\{$ S-groups $\}$. A group homomorphism $\alpha : H \longrightarrow G$ is called an (R,S)-homomorphism $\{$ S-homomorphism $\}$ if for all $h \in H$, $(hr)\alpha = (h\alpha)r$ for all $r \in (R,S)$ $\{ (hs)\alpha = (h\alpha)s$, for all $s \in S \}$.

Definition 1.6.6. A subset X of an (R,S) group G is called a free (R,S)-generating set or a basis of G if for every (R,S)-group H and for every mapping $\alpha : X \longrightarrow H$ there exists a unique (R,S)-homomorphism $\phi : G \longrightarrow H$ extending α, and such a group is called a free (R,S)-group.

The next result shows that for any non empty set X we can construct a free (R,S)-group with X as its basis.

Theorem 1.6.7. [11] Let $(R,S) \cong (Fr(S),S) / I$ be a d.g. near-ring and X be any set. Let $Fr(X,S)$ be the free group generated by the set $\{ x , s_x : x \in X , s \in S \}$ and

$$Fr(X,S)\, I = Gp\{ ga : g \in Fr(X,S) , a \in I \} .$$

Then $Fr(X,R,S) = Fr(X,S) / \overline{Fr(X,S)\, I}$ is the free (R,S)-group on X, where $\overline{Fr(X,S)\, I}$ is the normal closure of $Fr(X,S)\, I$ in $Fr(X,S)$.

Definition 1.6.8. Let (R,S) be a d.g. near-ring and G an (R,S)-group with representation θ. Then

1) the representation θ is called faithful if $\operatorname{Ker}\theta = \{0\}$,
2) if (R,S) has an identity element 1, the representation θ is said to be unitary and G a unitary (R,S)-group if θ maps 1 onto the identity endomorphism of G.

Theorem 1.6.9. [11] Let (R,S) be a d.g. near-ring with a left identity. Then (R,S) has a faithful representation on $(R,+)$.

Theorem 1.6.10 [11] If a d.g. near-ring has a faithful representation, then it has a faithful representation on $Fr(x,R,S)$, the free (R,S)-group on one generator.

We call a d.g. near-ring faithful if it has a faithful representation. J.D.P. Meldrum [11] constructed examples of d.g. near-rings which are not faithful.

Theorem 1.6.11. Let G be an (R,S)-group. If for every non zero $r \in (R,S)$ there is an element $g = g_r \in G$ such that $gr \neq 0$, then (R,S) has a faithful representation on G.

Theorem 1.6.12. [11] Let (R,S) be a faithful d.g. near-ring and $G = Fr(x,R,S)$, the free (R,S)-group on one generator. Then $G = G_1 * G_2$, the free product of G_1 and G_2, where G_1 is the free cyclic group and G_2 is a group isomorphic with $(R,+)$.

Definition 1.6.13 Let G be an (R,S)-group, (R,S) a d.g. near-ring. An (R,S)-subgroup H of G is called a submodule of G if it is normal in G.

Definition 1.6.14. Let (R,S) be a d.g. near-ring and $\mathcal{C}_{(R,S)}$ be the category of all (R,S)-groups. Then the coproduct object in $\mathcal{C}_{(R,S)}$ of a family of objects in $\mathcal{C}_{(R,S)}$ is called the free (R,S)-product of that family.

Theorem 1.6.15. [12] Let $\{ H_\lambda : \lambda \in \Lambda \}$ be a family of (R,S)-groups with $\{ X_\lambda : \lambda \in \Lambda \}$ as their (R,S)-generating sets. Then $Fr(X,R,S) / K$ is the free (R,S)-product of $\{ H_\lambda : \lambda \in \Lambda \}$, where X is the disjoint union of the family $\{ X_\lambda : \lambda \in \Lambda \}$ of sets and K is the submodule generated by $\{ \operatorname{Ker} \theta_\lambda : \lambda \in \Lambda \}$ for the natural homomorphisms $\theta_\lambda : Fr(X_\lambda, R, S) \longrightarrow H_\lambda$, $\lambda \in \Lambda$.

Chapter 2

Categories and D.G. Near-Rings

2.1. The Upper and Lower Faithful D.G. Near-Rings

The lower faithful d.g. near-ring for a d.g. near-ring (R,S) is a faithful d.g. near-ring $(\underline{R},\underline{S})$ with a d.g. near-ring epimorphism $\theta : (R,S) \longrightarrow (\underline{R},\underline{S})$ such that

1) $S\theta = \underline{S}$

2) if $\psi : (R,S) \longrightarrow (T,U)$ is a d.g. near-ring homomorphism, where (T,U) is a faithful d.g. near-ring, then there exists a unique d.g. near-ring homomorphism $\phi : (\underline{R},\underline{S}) \longrightarrow (T,U)$ such that $\theta\phi = \psi$.

The upper faithful d.g. near-ring for a d.g. near-ring (R,S) is a faithful d.g. near-ring $(\overline{R},S)$ with a d.g. near-ring epimorphism $\theta : (\overline{R},S) \longrightarrow (R,S)$ such that

1) θ / S = identity,

2) if ψ is any d.g. near-ring homomorphism from a faithful d.g. near-ring (T,U) to (R,S) then there exists a unique d.g. near-ring homomorphism $\phi : (T,U) \longrightarrow (\overline{R},S)$ such that $\phi\theta = \psi$.

Now we will show that for any d.g. near-ring (R,S) upper and lower faithful d.g. near-rings exist. First we prove the existence of lower faithful d.g. near-rings.

Theorem 2.1.1. Let (R,S) be a d.g. near-ring and

$$A = A(R,S) = \{ r \in (R,S) : Gr = 0 \},$$

where $G = Fr(x,R,S)$, the free (R,S)-group on one generator x . Then the quotient d.g. near-ring $(R,S) / A$ is the lower faithful d.g. near-ring for (R,S) .

We prove this theorem in an exactly similar way to theorem 4.3 [11]. So we need the following two lemmas of the same paper.

Lemma 2.1.2. Let $\theta : (R,S) \longrightarrow (T,U)$ be a d.g. near-ring homomorphism and G a (T,U)-group with representation ϕ . Then G is an (R,S)-group and the kernel of the representation μ of (R,S) on G is the inverse image under θ of $\mathrm{Ker}\phi$.

Proof. $\theta\phi$ is a d.g. near-ring homomorphism from (R,S) to $(E(G),End(G))$. Therefore $\mu = \theta\phi$ is a representation of (R,S) on G . Let $r \in \mathrm{Ker}\,\mu$ for some $r \in (R,S)$. Then $0 = r\mu = r\theta\phi$, i.e. , $r \in [\mathrm{Ker}\phi]\theta^{-1}$. Therefore $\mathrm{Ker}\,\mu \subseteq [\mathrm{Ker}\phi]\theta^{-1}$. On the other hand if $r \in [\mathrm{Ker}\phi]\theta^{-1}$, then $r\theta \in \mathrm{Ker}\phi$ and $r\mu = r\theta\phi = 0$. This shows that $[\mathrm{Ker}\phi]\theta^{-1} \subseteq \mathrm{Ker}\mu$. Hence we get $\mathrm{Ker}\,\mu = [\mathrm{Ker}\phi]\theta^{-1}$.

Lemma 2.1.3. Let (R,S) be a d.g. near-ring and $G = Fr(x,R,S)$ be the free (R,S)-group on one generator x . Let

$$A = A(R,S) = \{ r \in (R,S) : Gr = 0 \} .$$

If ϕ is any representation of (R,S) then $\mathrm{Ker}\phi$ contains A .

Proof. Let H be an (R,S)-group with representation ϕ .

Suppose that $\mathrm{Ker}\,\phi \not\supset A$, and let $r \in A - \mathrm{Ker}\,\phi$. Then there exists an $h \in H$ such that $h(r\phi) \neq 0$. Map x to this $h \in H$. Since G is the free (R,S)-group on one generator, this map extends to an (R,S)-homomorphism $\alpha : G \longrightarrow H$. Therefore, as $xr = 0$ in G,

$$0 = (xr)\alpha = (x\alpha)r\phi = h(r\phi)$$

in H . This is a contradiction. Hence $\mathrm{Ker}\phi$ contains A .

Proof of Theorem 2.1.1. Since A is an ideal of (R,S) we get a quotient d.g. near-ring $(R,S)/A$, which is faithful having a faithful representation on G . We denote $(R,S)/A$ by $(\underline{R},\underline{S})$. Let θ be the natural homomorphism from (R,S) to $(\underline{R},\underline{S})$. Then $\underline{S} = \{ s + A : s \in S \}$, i.e. , $S\theta = \underline{S}$. Note that θ/S need not be the identity mapping.

Now let ψ be a d.g. near-ring homomorphism from (R,S) to a faithful d.g. near-ring (T,U) and $H = Fr(x,T,U)$ be the free (T,U)-group on one generator x . By lemma 2.1.2 H is an (R,S)-group with representation $\psi\eta$, where η is the representation of (T,U) on H . Since (T,U) is faithful, by 1.6.10 it has a faithful representation on H . Therefore $\mathrm{Ker}\,\eta = \{0\}$, and we get $\mathrm{Ker}\,\psi\eta = \mathrm{Ker}\,\psi$. By lemma 2.1.3 , $A \subseteq \mathrm{Ker}\,\psi\eta = \mathrm{Ker}\,\psi$. So there exists a unique d.g. near-ring homomorphism $\phi : (\underline{R},\underline{S}) \longrightarrow (T,U)$ such that $\theta\phi = \psi$, i.e. , the following diagram is commutative.

$$
\begin{tikzcd}
(R,S) \arrow[r, "\theta"] \arrow[dr, "\psi"'] & (\underline{R},\underline{S}) \arrow[d, dashed, "\phi"] \\
& (T,U)
\end{tikzcd}
$$

This completes the proof.

Let (R,S) be a d.g.near-ring and $G = Gp\{x\} * R_x$ be the free product of the free group $Gp\{x\}$ on one generator and the group $R_x \cong (R,+)$. Define for $s \in S$ a map $\bar{s} : G \longrightarrow G$ by

$$x \longmapsto s_x \quad \text{and} \quad r_x \longmapsto (r\,s)_x \;.$$

Then since S is a distributive semigroup of (R,S), each $\bar{s}$ is an endomorphism of R_x. Therefore by the property of free products each $\bar{s}$ extends to an endomorphism of G, again denoted by $\bar{s}$.
Now the map $\beta : s \longmapsto \bar{s}$ is a semigroup homomorphism from S to $\mathrm{End}(G)$, for

$$x\,\bar{s}\,\bar{t} = (x\,\bar{s})\,\bar{t} = (s_x)\,\bar{t} = (s\;t)_x = x\,\overline{s\,t}$$

$$r_x\,\bar{s}\,\bar{t} = (r_x\,\bar{s})\,\bar{t} = (r\;s)_x\,\bar{t} = (r\;s\;t)_x = r_x\,\overline{s\,t}$$

$\forall\, s, t \in S$. Moreover β is injective, for $s \neq t$ in S implies that $s_x \neq t_x$ in R_x and $x\,\bar{s} = (s_x) \neq t_x = x\,\bar{t}$, i.e., $\bar{s} \neq \bar{t}$ in $\mathrm{End}(G)$. So we identify $\bar{S}$ with S and say that S

is a semigroup of endomorphisms of G . This S generates a d.g. near-ring $(\overline{R},S)$ which has a faithful representation on G . Now we prove the following theorem .

Theorem 2.1.4. $(\overline{R},S)$ is the upper faithful d.g. near-ring for (R,S) .

Proof. Let I_S be the obvious semigroup isomorphism from S contained in $\overline{R}$ to S contained in R , and let

$$\epsilon_1 s_1 + - - - - - - - - + \epsilon_n s_n = 0$$

in $(\overline{R},+)$. Then $0 = x(\epsilon_1 s_1 + - - - - - - + \epsilon_n s_n)$ in G , $x \in X$,

$$= \epsilon_1 s_{1_x} + - - - - - - - + \epsilon_n s_{n_x} \quad ,$$

$$= (\epsilon_1 s_1 + - - - - - + \epsilon_n s_n)_x \quad ,$$

$$= r_x \ ,$$

$r \in (R,+)$. But as $(R_x, +) \cong (R,+)$, $r_x = 0$ in $(R_x ,+)$ if and only if $r = 0$ in $(R,+)$. So $\epsilon_1 s_1 + - - - - - - - + \epsilon_n s_n = 0$ in $(R,+)$. So I_S extends to a group homomorphism θ from $(\overline{R},+)$ to $(R,+)$. By 1.5.13 θ is a d.g. near-ring homomorphism from $(\overline{R},S)$ to (R,S) . Clearly θ is onto and θ/ S = identity.

Now let (T,U) be a faithful d.g. near-ring with a d.g. near-ring homomorphism $\psi : (T,U) \longrightarrow (R,S)$. Then by the homomorphism theorem for d.g. near-rings we have

$$(T\psi, U\psi) \cong (T,U)/I ,$$

where $I = \operatorname{Ker}\psi$ is an ideal of (T,U). Let $H = Fr(x,T,U)$ be free (T,U)-group on one generator x. Since (T,U) is faithful, by 1.6.12, $H = Gp\{x\} * T_x$, the free product of the free group $Gp\{x\}$ and $T_x \cong (T,+)$. The homomorphism $\psi : (T,+) \longrightarrow (R,+)$ induces a homomorphism $\psi_x : t_x \longmapsto (t\psi)_x$ from T_x to R_x. Clearly $\operatorname{Ker}\psi_x = I_x$ and $T_x\psi_x = (T\psi)_x \cong (T/I, +)$. We define a map $\psi' : H \longrightarrow G$ by

$$x \longmapsto x \quad \text{and} \quad t_x \longmapsto (t\psi)_x .$$

By the property of free products this map extends to a homomorphism $\bar{\psi} : H \longrightarrow G$, such that

$$\bar{\psi} / T_x = \psi_x \quad \text{and} \quad \bar{\psi} / Gp\{x\} = \text{identity}.$$

Now since $I_x U \subseteq (I\,U)_x$ and I is an ideal of (T,U), we get

$$I_x U \subseteq I_x .$$

This implies that I_x is a U-group. Therefore the normal closure I_x^H of I_x in H is also a U-group and hence by 1.6.3, it is a (T,U)-group. Thus H / I_x^H is a (T,U)-group and we get

$$\begin{aligned} H / I_x^H &\cong Gp\{x\} * T_x / I_x \qquad (1.3.5), \\ &\cong Gp\{x\} * (T\psi)_x , \\ &= G' \subseteq G = Gp\{x\} * R_x . \end{aligned}$$

Thus we have proved that G contains a subgroup $G' \cong H / I_x^H$

which is a (T,U)-group . The action of U on H / I_X^H is the same as that of $U\psi$ on G'. Therefore the semigroup homomorphism

$$\psi : U \subseteq T \longrightarrow S \subseteq \overline{R}$$

extends to a group homomorphism $\phi : (T,+) \longrightarrow (\overline{R},+)$. Again by 1. 5.13 , ϕ is a d.g. near-ring homomorphism from (T,U) to $(\overline{R},S)$ such that $\phi / U = \psi$. Also for $u \in U$,

$$u\phi\,\theta = u\psi\,\theta = u\psi \ .$$

This shows that $\phi\theta$ and ψ agree on U . So they do so on (T,U) . Hence $\phi\,\theta = \psi$ as d.g. near-ring homomorphisms . Clearly ϕ is unique with this property .

From now on we will denote by $\mathcal{A}$ the category of all d.g. near-rings and by $\mathcal{B}$ the category of all faithful d.g. near-rings. Then $\mathcal{B}$ is a full subcategory of $\mathcal{A}$. From § 2.1 we see that the lower faithful d.g. near-ring $(\underline{R},\underline{S})$ for (R,S) defines a coreflection in $\mathcal{B}$ of $(R,S) \in \mathcal{A}$ and since every d.g. near-ring in $\mathcal{A}$ has a lower faithful d.g. near-ring associated with it , $\mathcal{B}$ is a coreflective subcategory of $\mathcal{A}$. By denoting $(\underline{R},\underline{S}) = (R,S)\,F$ we get a covariant functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ which is a coadjoint of the inclusion functor $I : \mathcal{B} \longrightarrow \mathcal{A}$ (1.1.29) .

Similarly the upper faithful d.g. near-ring $(\overline{R},S)$ defines a reflection of $(R,S) \in \mathcal{A}$ in $\mathcal{B}$ and $\mathcal{B}$ is a reflective subcategory of $\mathcal{A}$ since each $(R,S) \in \mathcal{A}$ has a reflection $(\overline{R},S)$ in $\mathcal{B}$. Again by denoting $(\overline{R},S) = (R,S)G$ we get a covariant functor $G : \mathcal{A} \longrightarrow \mathcal{B}$ which is an adjoint of the inclusion functor $I : \mathcal{B} \longrightarrow \mathcal{A}$ (1.1.29).

We now go on to show that the categories $\mathcal{A}$ and $\mathcal{B}$ are complete and cocomplete. But since $\mathcal{B}$ is a reflective subcategory of $\mathcal{A}$, by theorems 1.1.30 and 1.1.31, it is sufficient to prove the existence of limits and colimits in $\mathcal{A}$.

2.2. Colimits in $\mathcal{A}$

First we prove the existence of special colimits in $\mathcal{A}$, namely the coproducts. Let $\{ (R_\lambda,S_\lambda) : \lambda \in \Lambda \}$ be a family of d.g. near-rings in $\mathcal{A}$. For each $\lambda \in \Lambda$, we have the free d.g. near-ring $F_\lambda = (Fr(S_\lambda),S_\lambda)$ such that $F_\lambda / A_\lambda \cong (R_\lambda,S_\lambda)$ for some ideal A_λ of F_λ. Let $S^* = \underset{\lambda \in \Lambda}{*} S_\lambda$ be the free product of the family $\{ S_\lambda : \lambda \in \Lambda \}$ of semigroups. Then we have another free d.g. near-ring $(Fr(S^*),S^*)$. Let $\{ u_\lambda : S_\lambda \longrightarrow S^* \}_{\lambda \in \Lambda}$ be the family of semigroup monomorphisms. By the property of free groups each u_λ extends to a group monomorphism from $(Fr(S_\lambda),+)$ to $(Fr(S^*),+)$ again denoted by u_λ. Hence each u_λ is a d.g. near-ring homomorphism from $(Fr(S_\lambda),S_\lambda)$ to $(Fr(S^*),S^*)$ (1.5.13). Since the free d.g. near-rings are unique up to d.g. near-ring isomorphism (1.5.14), for each $\lambda \in \Lambda$ identifying $(Fr(S_\lambda), S_\lambda)$ with its isomorphic image in $(Fr(S^*), S^*)$, we can consider it as a sub d.g. near-ring of $(Fr(S^*), S^*)$. Now let A be the ideal of $(Fr(S^*), S^*)$ generated by $\{ A_\lambda : \lambda \in \Lambda \}$. Then we prove the following theorem on similar lines to 1.1 [12].

Theorem 2.2.1. $\{ \alpha_\lambda : (R_\lambda, S_\lambda) \longrightarrow (Fr(S^*), S^*)/A \}_{\lambda \in \Lambda}$ is the coproduct of $\{ (R_\lambda, S_\lambda) : \lambda \in \Lambda \}$ in $\mathcal{R}$.

Proof. For each $\lambda \in \Lambda$, $A \cap F_\lambda$ is an ideal of $(Fr(S_\lambda), S_\lambda)$ containing A_λ. So there exists a unique d.g. near-ring homomorphism $\beta_\lambda : F_\lambda / A_\lambda \longrightarrow F_\lambda / A \cap F_\lambda$ such that the diagram

$$\begin{tikzcd}
F_\lambda \arrow[rr, "\pi_\lambda"] \arrow[d, "\gamma_\lambda"'] & & F_\lambda / A_\lambda \arrow[dll, dashed, "\beta_\lambda"] \\
F_\lambda / A \cap F_\lambda & &
\end{tikzcd}$$

where π_λ and γ_λ are natural homomorphisms.

commutes for each $\lambda \in \Lambda$. By the isomorphism theorem for d.g. near-rings we get $F_\lambda / A \cap F_\lambda \cong F_\lambda + A / A$, for each $\lambda \in \Lambda$. Thus we get a d.g. near-ring homomorphism $\alpha_\lambda : (R_\lambda, S_\lambda) \longrightarrow (Fr(S^*), S^*)/A$ for each $\lambda \in \Lambda$, which is a composition of the following previously defined homomorphisms

$$(R_\lambda, S_\lambda) \longrightarrow F_\lambda / A_\lambda \longrightarrow F_\lambda / A \cap F_\lambda \longrightarrow F_\lambda + A / A \longrightarrow F / A,$$

where $F / A = (Fr(S^*), S^*) / A$. For each $\lambda \in \Lambda$, α_λ maps

$$s_\lambda \longmapsto s_\lambda + A_\lambda \longmapsto s_\lambda + A \cap F_\lambda \longmapsto s_\lambda + A \quad , \; s_\lambda \in S_\lambda .$$

Let (T,U) be any d.g. near-ring in $\mathcal{A}$ together with a family $\{ \phi_\lambda : (R_\lambda, S_\lambda) \longrightarrow (T,U) \}_{\lambda \in \Lambda}$ of d.g. near-ring homomorphisms. This gives us a family $\{ \phi_\lambda : S_\lambda \longrightarrow U \}_{\lambda \in \Lambda}$ of semigroup homomorphisms. Since $\{ u_\lambda : S_\lambda \longrightarrow S^* \}_{\lambda \in \Lambda}$ is the free product of $\{ S_\lambda : \lambda \in \Lambda \}$, there exists a unique semigroup homomorphism $\theta : S^* \longrightarrow U$ such that $u_\lambda \theta = \phi_\lambda$ for each $\lambda \in \Lambda$. Again by the property of free groups θ extends to a group homomorphism $(Fr(S^*), +) \longrightarrow (T,+)$, again denoted by θ. Hence θ is a d.g. near-ring homomorphism from $(Fr(S^*),S^*)$ to (T,U). Now for each $\lambda \in \Lambda$

$$\begin{aligned}(Fr(S_\lambda),+)\,\theta &= Gp\{ S_\lambda \theta \} , \\ &= Gp\{ S_\lambda \phi_\lambda \} .\end{aligned}$$

Therefore $\qquad (Fr(S_\lambda),+)\,\theta = (R_\lambda, +)\,\phi_\lambda$, for each $\lambda \in \Lambda$.

But $(R_\lambda, S_\lambda) \cong (Fr(S_\lambda), S_\lambda) / A_\lambda = F_\lambda / A_\lambda$. This shows that $A_\lambda \subseteq \operatorname{Ker} \theta$ for each $\lambda \in \Lambda$, whence $A \subseteq \operatorname{Ker} \theta$. Thus we get a unique d.g. near-ring homomorphism $\phi : (Fr(S^*),S^*) / A \longrightarrow (T,U)$ such that $\pi \phi = \theta$, where π is the natural homomorphism from $(Fr(S^*),S^*)$ to $(Fr(S^*),S^*) / A$. Now we consider the following diagram

$$\begin{tikzcd}
(R_\lambda, S_\lambda) \arrow[r, "\alpha_\lambda"] \arrow[d, "\phi_\lambda"'] & (Fr(S^*), S^*) / A \arrow[dl, "\phi"] \\
(T,U) &
\end{tikzcd}$$

For each $\lambda \in \Lambda$ and $s_\lambda \in S_\lambda$, we have from above

$$s_\lambda \alpha_\lambda \phi = (s_\lambda + A)\phi = s_\lambda \pi \phi = s_\lambda \theta = s_\lambda \phi_\lambda .$$

This shows that $\alpha_\lambda \phi$ and ϕ_λ agree on the generators of (R_λ, S_λ), so they do so on (R_λ, S_λ). Hence $\alpha_\lambda \phi = \phi_\lambda$ for each $\lambda \in \Lambda$, i.e. the above diagram is commutative for each $\lambda \in \Lambda$. Moreover we show that ϕ is unique with this property. For that, let ψ be another d.g. near-ring homomorphism from $(Fr(S^*), S^*) / A$ to (T,U) such that $\alpha_\lambda \psi = \phi_\lambda$ for each $\lambda \in \Lambda$. Then

$$s_\lambda \alpha_\lambda \psi = s_\lambda \phi_\lambda = s_\lambda \alpha_\lambda \phi$$

or $$(s_\lambda + A)\psi = (s_\lambda + A)\phi .$$

Since the set $\{ s_\lambda + A \ : s_\lambda \in S_\lambda , \lambda \in \Lambda \}$ generates $(Fr(S^*), S^*)/A$ and ψ and ϕ agree on this set, therefore they agree on $(Fr(S^*), S^*)/A$. Hence $\psi = \phi$.

To complete the proof we need to show that α_λ, for each $\lambda \in \Lambda$, is a monomorphism. For that let $(T,U) = (R_\mu, S_\mu)$ for $\mu \in \Lambda$ and

$$\phi_\lambda = \delta_{\lambda\mu}\text{, where } \delta_{\lambda\mu} = 1 \text{ if } \lambda = \mu$$

$$= 0 \text{ if } \lambda \neq \mu .$$

Therefore we get $\alpha_\lambda \phi = \delta_{\lambda\mu}$, for each $\lambda \in \Lambda$, and hence $\alpha_\mu \phi = I_{(R_\mu, S_\mu)}$ for each $\mu \in \Lambda$. Therefore by 1.1.6 α_μ is a monomorphism for each $\mu \in \Lambda$.

Now we prove the existence of general colimits in $\mathcal{A}$. Let D be a diagram in $\mathcal{A}$ over a scheme (Λ, M, d). By theorem 2.2.1 the coproduct $\underset{\lambda \in \Lambda}{*} (R_\lambda, S_\lambda)$ of the family $\{ (R_\lambda, S_\lambda) : \lambda \in \Lambda \}$ of d.g. near-rings involved in D exists in $\mathcal{A}$ with the family $\{ \alpha_\lambda : (R_\lambda, S_\lambda) \longrightarrow \underset{\lambda \in \Lambda}{*} (R_\lambda, S_\lambda) \}_{\lambda \in \Lambda}$ of d.g. near-ring monomorphisms. Let K be the ideal of $\underset{\lambda \in \Lambda}{*} (R_\lambda, S_\lambda)$ generated by $\cup \{ \text{Image}(\alpha_\lambda - m_{\lambda\mu}\alpha_\mu) : m \in M \}$. Then we get a quotient d.g. near-ring $\underset{\lambda \in \Lambda}{*} (R_\lambda, S_\lambda)/K$ with natural homomorphism π from $\underset{\lambda \in \Lambda}{*} (R_\lambda, S_\lambda)$ to $\underset{\lambda \in \Lambda}{*} (R_\lambda, S_\lambda)/K$. We have

$$\text{Image}(\alpha_\lambda \pi - m_{\lambda\mu}\alpha_\mu \pi) = [\text{Image}(\alpha_\lambda - m_{\lambda\mu}\alpha_\mu)]\pi ,$$

$$= 0 .$$

Therefore the family $\{ \alpha_\lambda \pi : (R_\lambda, S_\lambda) \longrightarrow \underset{\lambda \in \Lambda}{*} (R_\lambda, S_\lambda)/K \}_{\lambda \in \Lambda}$ is cocompatible for D, i.e. for each $m \in M$ we have a commutative diagram

$$\begin{tikzcd}
(R_\lambda, S_\lambda) \arrow[dd, "m_{\lambda\mu}"'] \arrow[dr, "\alpha_\lambda"] & & \\
& \underset{\lambda \in \Lambda}{*} (R_\lambda, S_\lambda) \arrow[r, "\pi"] & \underset{\lambda \in \Lambda}{*} (R_\lambda, S_\lambda)/ K \\
(R_\mu, S_\mu) \arrow[ur, "\alpha_\mu"'] & &
\end{tikzcd}$$

Then we prove the following theorem :

Theorem 2.2.2. $\{ \alpha_\lambda \pi : (R_\lambda, S_\lambda) \longrightarrow \underset{\lambda \in \Lambda}{*} (R_\lambda, S_\lambda)/ K \}$ is a colimit of the diagram D in $\mathcal{A}$ over a scheme (Λ, M, d) .

Proof. We have seen that $\{ \alpha_\lambda \pi : \lambda \in \Lambda \}$ is a cocompatible family of d.g. near-ring homomorphisms for D . Let (T,U) be a d.g. near-ring in $\mathcal{A}$ with a cocompatible family $\{ f_\lambda : (R_\lambda, S_\lambda) \longrightarrow (T,U) \}_{\lambda \in \Lambda}$ of d.g. near-ring homomorphisms for D . By the property of coproducts there exists a unique d.g. near-ring homomorphism $\phi : \underset{\lambda \in \Lambda}{*} (R_\lambda, S_\lambda) \longrightarrow (T,U)$ such that for each $\lambda \in \Lambda$ we have a commutative diagram

$$\begin{tikzcd}
(R_\lambda, S_\lambda) \arrow[r, "\alpha_\lambda"] \arrow[dr, "f_\lambda"'] & \underset{\lambda \in \Lambda}{*} (R_\lambda, S_\lambda) \arrow[d, dashed, "\phi"] \\
& (T,U)
\end{tikzcd}$$

Now for each $m \in M$, we have

$$\{ \mathrm{Image}(\alpha_\lambda - m_{\lambda\mu}\alpha_\mu) \}\phi = \mathrm{Image}(\alpha_\lambda\phi - m_{\lambda\mu}\alpha_\mu\phi)$$

$$= \mathrm{Image}(f_\lambda - m_{\lambda\mu}f_\mu)$$

By the cocompatibility of $\{ f_\lambda : \lambda \in \Lambda \}$ for D ,

$$\mathrm{Image}(f_\lambda - m_{\lambda\mu}f_\mu) = 0 .$$

Therefore $\mathrm{Image}(\alpha_\lambda - m_{\lambda\mu}\alpha_\mu) \subseteq \mathrm{Ker}\,\phi$, for each $m \in M$, and hence $K \subseteq \mathrm{Ker}\,\phi$. So there exists a unique d.g. near-ring homomorphism $\psi : \underset{\lambda \in \Lambda}{*}(R_\lambda, S_\lambda)/K \longrightarrow (T,U)$ such that we have a commutative diagram

$$\begin{tikzcd}
\underset{\lambda \in \Lambda}{*}(R_\lambda, S_\lambda) \arrow[r, "\pi"] \arrow[dr, "\phi"'] & \underset{\lambda \in \Lambda}{*}(R_\lambda, S_\lambda)/K \arrow[d, dashed, "\psi"] \\
& (T,U)
\end{tikzcd}$$

Now for each $\lambda \in \Lambda$, we have from above diagrams ,

$$(\alpha_\lambda\pi)\psi = \alpha_\lambda(\pi\psi) ,$$

$$= \alpha_\lambda\phi ,$$

$$= f_\lambda .$$

Finally, we prove that ψ is unique with this property. For that let $\psi' : \underset{\lambda \in \Lambda}{*}(R_\lambda, S_\lambda) \longrightarrow (T,U)$ be another d.g. near-ring homomorphism such that $(\alpha_\lambda\pi)\psi' = f_\lambda$ for each $\lambda \in \Lambda$. Then from the uniqueness of ϕ with $\alpha_\lambda \phi = f_\lambda$ for each $\lambda \in \Lambda$, we get $\pi \psi' = \phi$ and then from the uniqueness of ψ such that $\pi \psi = \phi$, we get $\psi' = \psi$. This completes the proof.

2.3. Limits in $\mathcal{A}$

First we look for the products in $\mathcal{A}$. Let $\{ (R_\lambda, S_\lambda) : \lambda \in \Lambda \}$ be a family of d.g. near-rings in $\mathcal{A}$, and let $Q = \prod_{\lambda \in \Lambda} R_\lambda$ be the cartesian product of the family $\{ R_\lambda : \lambda \in \Lambda \}$ considered as near-rings. Then Q is a near-ring which is not necessarily d.g. Let $S = \prod_{\lambda \in \Lambda} S_\lambda$ be the cartesian product of the family $\{ S_\lambda : \lambda \in \Lambda \}$ of semigroups. Then S is a sub-semigroup of Q. Now for all $(r_\lambda)_{\lambda \in \Lambda}$, $(r'_\lambda)_{\lambda \in \Lambda}$ belonging to Q, and for all $(s_\lambda)_{\lambda \in \Lambda}$ belonging to S we have

$$\begin{aligned}
\{ (r_\lambda)_{\lambda \in \Lambda} + (r'_\lambda)_{\lambda \in \Lambda} \} (s_\lambda)_{\lambda \in \Lambda} &= (r_\lambda + r'_\lambda)_{\lambda \in \Lambda} (s_\lambda)_{\lambda \in \Lambda} , \\
&= ((r_\lambda + r'_\lambda) s_\lambda)_{\lambda \in \Lambda} , \\
&= (r_\lambda s_\lambda + r'_\lambda s_\lambda)_{\lambda \in \Lambda} , \\
&= (r_\lambda s_\lambda)_{\lambda \in \Lambda} + (r'_\lambda s_\lambda)_{\lambda \in \Lambda} , \\
&= (r_\lambda)_{\lambda \in \Lambda} (s_\lambda)_{\lambda \in \Lambda} + (r'_\lambda)_{\lambda \in \Lambda} (s_\lambda)_{\lambda \in \Lambda} .
\end{aligned}$$

This shows that the semigroup S is distributive in Q. This S generates a sub d.g. near-ring (R, S) of Q. Let $\{ p_\lambda : Q \longrightarrow R_\lambda \}_{\lambda \in \Lambda}$ be the near-ring projection homomorphisms.

Then for each $\lambda \in \Lambda$, p_λ maps S onto S_λ and hence $(R,+)$ onto $(R_\lambda,+)$. Therefore , for each $\lambda \in \Lambda$, $q_\lambda = p_\lambda / (R,S)$ is a d.g. near-ring epimorphism onto (R_λ, S_λ) . Now we prove the following theorem :

Theorem 2.3.1. $\{ q_\lambda : (R,S) \longrightarrow (R_\lambda, S_\lambda) \}_{\lambda \in \Lambda}$ is the product of $\{ (R_\lambda, S_\lambda) : \lambda \in \Lambda \}$ in $\mathcal{A}$.

Proof. Let (T,U) be a d.g. near-ring in $\mathcal{A}$ with a family $\{ \phi_\lambda : (T,U) \longrightarrow (R_\lambda, S_\lambda) \}_{\lambda \in \Lambda}$ of d.g. near-ring homomorphisms. Then we have a family $\{ \phi_\lambda : U \longrightarrow S_\lambda \}_{\lambda \in \Lambda}$ of semigroup homomorphisms and since $\{ q_\lambda : S \longrightarrow S_\lambda \}_{\lambda \in \Lambda}$ is the product of $\{ S_\lambda : \lambda \in \Lambda \}$, we get , for each $\lambda \in \Lambda$, a commutative diagram

$$\begin{tikzcd} S \arrow[rr, "q_\lambda"] & & S_\lambda \\ & U \arrow[ul, "\phi"] \arrow[ur, "\phi_\lambda"'] & \end{tikzcd}$$

for a uniquely defined semigroup homomorphism ϕ . Now let $\epsilon_1 u_1 + \text{--------} + \epsilon_n u_n = 0$ in (T,U) . Then, for each $\lambda \in \Lambda$, $(\epsilon_1 u_1 + \text{------} + \epsilon_n u_n)\phi_\lambda = 0$ in (R_λ, S_λ) . Therefore , for each $\lambda \in \Lambda$, we have in (R_λ, S_λ) ,

$$0 = \epsilon_1 u_1 \phi_\lambda + \text{----------} + \epsilon_n u_n \phi_\lambda$$

$$= \epsilon_1 u_1 \phi\, q_\lambda + \text{-------} + \epsilon_n u_n \phi\, q_\lambda$$

$$= (\epsilon_1 u_1 \phi + \text{--------} + \epsilon_n u_n \phi)\, q_\lambda \ .$$

Hence $\epsilon_1 u_1 \phi + \text{------} + \epsilon_n u_n \phi = 0$ in Q, and since $\epsilon_1 u_1 \phi + \text{------} + \epsilon_n u_n \phi$ belongs to (R,S), ϕ extends to a group homomorphism from $(T,+)$ to $(R,+)$. We denote this homomorphism also by ϕ. Thus ϕ is a d.g. near-ring homomorphism from (T,U) to (R,S). Also we have $\phi\, q_\lambda = \phi_\lambda$, for each $\lambda \in \Lambda$, as d.g. near-ring homomorphisms. Moreover, the uniqueness of ϕ as a semigroup homomorphism with $\phi\, q_\lambda = \phi_\lambda$, for each $\lambda \in \Lambda$, gives us its uniqueness as a d.g. near-ring homomorphism with the same property.

Corollary 2.3.2. If $\{ q_\lambda : (R,S) \longrightarrow (R_\lambda, S_\lambda) \}_{\lambda \in \Lambda}$ is the product in $\mathcal{A}$ of $\{ (R_\lambda, S_\lambda) : \lambda \in \Lambda \}$, then $\{ q_\lambda : S \longrightarrow S_\lambda \}_{\lambda \in \Lambda}$ is the product of $\{ S_\lambda : \lambda \in \Lambda \}$ in the category of all semigroups.

Proof. If $\{ q_\lambda : (R,S) \longrightarrow (R_\lambda, S_\lambda) \}_{\lambda \in \Lambda}$ is the product in $\mathcal{A}$ of $\{ (R_\lambda, S_\lambda) : \lambda \in \Lambda \}$ then we have a family $\{ q_\lambda : S \longrightarrow S_\lambda \}_{\lambda \in \Lambda}$ of semigroup epimorphisms. Let U be a semigroup with a family $\{ f_\lambda : U \longrightarrow S_\lambda \}_{\lambda \in \Lambda}$ of semigroup homomorphisms. Then we get a family $\{ \overline{f}_\lambda : (Fr(U), U) \longrightarrow (R_\lambda, S_\lambda) \}_{\lambda \in \Lambda}$ of d.g. near-ring homomorphisms, where $(Fr(U), U)$ is the free d.g. near-ring on U and $\overline{f}_\lambda / U = f_\lambda$, for each $\lambda \in \Lambda$. From the hypothesis, there exists a unique d.g. near-ring homomorphism ϕ from $(Fr(U),U)$ to (R,S) such that $\phi q_\lambda = \overline{f}_\lambda$ for each $\lambda \in \Lambda$. Obviously $\phi q_\lambda = f_\lambda$ for each $\lambda \in \Lambda$ as semigroup homomorphisms, and ϕ from U to S is unique with this property. Hence $\{ q_\lambda : S \longrightarrow S_\lambda \}_{\lambda \in \Lambda}$ is the product of $\{ S_\lambda : \lambda \in \Lambda \}$ in the category of all semigroups.

Before proving the existence of general limits in $\mathcal{A}$, we give a result, from theory of categories, which we are going to use.

Theorem 2.3.3. [13] Let $\mathcal{C}$ be a category with products and finite intersections and let D be a diagram in $\mathcal{C}$ over a scheme (Λ, M, d). Then a limit for D is given by the compositions

$$\cap \{ \mathrm{Equ}\,(p_\lambda m D\, ,\, p_\mu) : m \in M \} \subseteq \prod_{\lambda \in \Lambda} A_\lambda \xrightarrow{\ p_\lambda\ } A_\lambda \ ,$$

where $md = (\lambda, \mu)$, $mD : A_\lambda \longrightarrow A_\mu$ and p_λ represents the λ-th projection.

Now let $\mathcal{S}$ be the category of all semigroups with zero element. Then in $\mathcal{S}$ any homomorphism $\alpha : U \longrightarrow V$ maps 0_U onto 0_V. Also $\mathcal{S}$ is a category with intersections and products, since $\cap \{ U_\lambda : \lambda \in \Lambda\, ,\, U_\lambda \in \mathcal{S} \}$ always contains the zero element and if $U = \prod_{\lambda \in \Lambda} U_\lambda$ then $(0_{U_\lambda})_{\lambda \in \Lambda}$ is the zero element of U.

Let D be a diagram in $\mathcal{A}$ over a scheme (Λ, M, d). By theorem 2.3.1 the product $\{ q_\lambda : (R,S) \longrightarrow (R_\lambda, S_\lambda) \}_{\lambda \in \Lambda}$ of the family $\{ (R_\lambda, S_\lambda) : \lambda \in \Lambda \}$ of d.g. near-rings involved in D, exists in $\mathcal{A}$. Without loss of generality we can assume that $0 \in S_\lambda$, for each $\lambda \in \Lambda$, i.e. we can consider each S_λ in $\mathcal{S}$.

By corollary 2.3.2, $\{ q_\lambda : S \longrightarrow S_\lambda \}_{\lambda \in \Lambda}$ is the product of the family $\{ S_\lambda : \lambda \in \Lambda \}$ of semigroups involved in the corresponding diagram D in $\mathcal{S}$. Let $S' = \cap \{ \operatorname{Equ}(q_\lambda m_{\lambda\mu}, q_\mu) : m \in M \} \subseteq S$. Then S' is not empty since $0 \in S'$. Let $q'_\lambda = q_\lambda / S'$ for each $\lambda \in \Lambda$. Then by theorem 2.3.3, $\{ q'_\lambda : S' \longrightarrow S_\lambda \}_{\lambda \in \Lambda}$ is a limit for D in $\mathcal{S}$. S' generates a sub d.g. near-ring (R', S') of (R, S) and q'_λ is a d.g. near-ring homomorphism from (R', S') to (R_λ, S_λ) for each $\lambda \in \Lambda$. Now we prove the following theorem :

Theorem 2.3.4. $\{ q_\lambda : (R', S') \longrightarrow (R_\lambda, S_\lambda) \}_{\lambda \in \Lambda}$ is a limit for D in $\mathcal{A}$.

Proof. First we prove that the family $\{ q'_\lambda : \lambda \in \Lambda \}$ is compatible for D, i.e. the following diagram is commutative for each $m \in M$.

$$\begin{tikzcd}
 & (R_\lambda, S_\lambda) \arrow[d, "m_{\lambda\mu}"] \\
(R', S') \arrow[ur, "q'_\lambda"] \arrow[dr, "q'_\mu"'] & \\
 & (R_\mu, S_\mu)
\end{tikzcd}$$

For all $r \in (R',S')$ and for all $m \in M$, we get from above :

$$\begin{aligned} r\, q'_\lambda m_{\lambda\mu} &= (\epsilon_1 s'_1 + \cdots + \epsilon_n s'_n)\, q'_\lambda m_{\lambda\mu} \,, \\ &= \epsilon_1 s'_1 q'_\lambda m_{\lambda\mu} + \cdots + \epsilon_n s'_n q'_\lambda m_{\lambda\mu} \,, \\ &= \epsilon_1 s'_1 q'_\mu + \cdots + \epsilon_n s'_n q'_\mu \,, \\ &= (\epsilon_1 s'_1 + \cdots + \epsilon_n s'_n)\, q'_\mu \,, \\ &= r\, q'_\mu \,, \end{aligned}$$

i.e. $q'_\lambda m_{\lambda\mu} = q'_\mu$ for each $m \in M$. Hence $\{ q'_\lambda : \lambda \in \Lambda \}$ is compatible for D in $\mathcal{A}$.

Let (T,U) be a d.g. near-ring with a family, compatible for D, $\{ \phi_\lambda : (T,U) \longrightarrow (R_\lambda, S_\lambda) \}_{\lambda \in \Lambda}$ of d.g. near-ring homomorphisms. Then by the property of products there exists a unique d.g. near-ring homomorphism $\phi : (T,U) \longrightarrow (R,S)$ such that $\phi\, q_\lambda = \phi_\lambda$ for each $\lambda \in \Lambda$. Since $\{ q'_\lambda : S' \longrightarrow S_\lambda \}_{\lambda \in \Lambda}$ is a limit in $\mathcal{S}$ for D, ϕ / U factors uniquely through S'. Therefore ϕ maps U into S'. Hence ϕ maps $(T,+)$ into $(R',+)$ so that ϕ factors through (R',S') as a d.g. near-ring homomorphism. Clearly this factorisation is unique. Hence we have proved the theorem.

Chapter 3

Surjective Reflections

Introduction

In some cases, proving the existence of limits, in a subcategory, may be easier than proving this in the bigger category; for example, the existence of products in $\mathcal{B}$, the category of faithful d.g. near-rings, can be proved in the following elementary way:

Let $\{(R_\lambda, S_\lambda) : \lambda \in \Lambda\}$ be a family of faithful d.g. near-rings. Let G_λ be a faithful (R_λ, S_λ)-group, for each $\lambda \in \Lambda$. Then we can show that $S = \prod_{\lambda \in \Lambda} S_\lambda$ is a semigroup of endomorphisms of the group $G = \prod_{\lambda \in \Lambda} G_\lambda$. This semigroup S generates a d.g. near-ring (R,S) inside $E(G)$. Each semigroup projection homomorphism $p_\lambda : S \longrightarrow S_\lambda$ extends to a group homomorphism $p_\lambda : (R, +) \longrightarrow (R_\lambda, +)$. It is easy to see that $\{ p_\lambda : (R,S) \longrightarrow (R_\lambda, S_\lambda) \}_{\lambda \in \Lambda}$ is the product in $\mathcal{B}$ of $\{(R_\lambda, S_\lambda) : \lambda \in \Lambda\}$.

As mentioned before, with the help of products and general limits in $\mathcal{B}$, we were able to prove the existence of products and limits in $\mathcal{A}$. The reason for this is that each reflection homomorphism in this case is surjective. In this chapter we generalize this method for all algebraic categories satisfying a certain condition, which holds in the case of d.g. near-rings.

3.1. Surjective Reflections

Let $\mathcal{C}$ be an algebraic category. Let $\{ \alpha_\lambda : C \longrightarrow C_\lambda \}_{\lambda \in \Lambda}$ be a set of morphisms in $\mathcal{C}$ and let

$$\rho_\lambda = \{ (x, y) \in C \times C : x\,\alpha_\lambda = y\,\alpha_\lambda \}$$

denote the kernel of α_λ, $\lambda \in \Lambda$. Suppose further that $\rho^{\natural} : C \longrightarrow C / \rho$ and each induced homomorphism $\overline{\alpha}_\lambda : C/\rho \longrightarrow C\,\alpha_\lambda$ is in $\mathcal{C}$, where $\rho = \cap \{ \rho_\lambda : \lambda \in \Lambda \}$. Note that $\rho \neq \phi$, since $(c,c) \in \rho_\lambda$ for each $c \in C$ and each $\lambda \in \Lambda$.

Let $\mathcal{D}$ be a reflective subcategory of $\mathcal{C}$ such that each reflection homomorphism $\theta_C : C\,T \longrightarrow C$ is surjective, T being the reflector functor. Suppose that for a diagram D in $\mathcal{D}$ over a scheme (Λ, M, d) a limit exists in $\mathcal{D}$ and consider the same diagram in $\mathcal{C}$. Let $\{ C_\lambda : \lambda \in \Lambda \}$ be the family of objects of $\mathcal{C}$ involved in D. Then we have the corresponding diagram D in $\mathcal{D}$, involving the family $\{ C_\lambda T : \lambda \in \Lambda \}$ of objects of $\mathcal{D}$, and for each $m \in M$, the following diagram is commutative,

$$\begin{tikzcd}
C_\lambda T \arrow[r, "\theta_\lambda"] \arrow[d, dashed, "m_{\lambda\mu} T"'] & C_\lambda \arrow[d, "m_{\lambda\mu}"] \\
C_\mu T \arrow[r, "\theta_\mu"] & C_\mu
\end{tikzcd}$$

where θ_λ's are reflection homomorphisms.

Let $\{ p_\lambda : C \longrightarrow C_\lambda T \}_{\lambda \in \Lambda}$ be a limit of the diagram D in $\mathcal{D}$. Put $q_\lambda = p_\lambda \theta_\lambda$ for each $\lambda \in \Lambda$. Then

$$\sigma_\lambda = \{(x, y) \in C \times C : x q_\lambda = y q_\lambda \}$$

is the kernel of q_λ for each $\lambda \in \Lambda$. Let $\sigma = \cap \{ \sigma_\lambda : \lambda \in \Lambda \}$. From above $\sigma^{\natural} : C \longrightarrow C/\sigma$ and each induced homomorphism $\phi_\lambda : C/\sigma \longrightarrow C_\lambda$ are in $\mathcal{C}$ and $\sigma^{\natural}\phi_\lambda = q_\lambda$, $\lambda \in \Lambda$. We claim that $\{ \phi_\lambda : C/\sigma \longrightarrow C_\lambda \}_{\lambda \in \Lambda}$ is a limit of D in $\mathcal{C}$. First we show that this family is compatible for D. For each $m \in M$ we have from above :

$$\sigma^{\natural}\phi_\lambda m_{\lambda\mu} = q_\lambda m_{\lambda\mu} = p_\lambda \theta_\lambda m_{\lambda\mu} = p_\lambda(m_{\lambda\mu}T)\, \theta_\mu = p_\mu \theta_\mu = q_\mu$$
$$= \sigma^{\natural}\phi_\mu .$$

Since $\sigma^{\natural}$ is a surjection, we get $\phi_\lambda m_{\lambda\mu} = \phi_\mu$, for all $m \in M$. Thus $\{ \phi_\lambda : C/\sigma \longrightarrow C_\lambda \}_{\lambda \in \Lambda}$ is compatible for D. Now let $\{ f_\lambda : A \longrightarrow C_\lambda \}_{\lambda \in \Lambda}$ be any compatible family for D, in $\mathcal{C}$. Then we get a family $\{ f_\lambda T : AT \longrightarrow C_\lambda T \}_{\lambda \in \Lambda}$ in $\mathcal{D}$, which is compatible for D, because for all $m \in M$ we have :

$$(f_\lambda T)(m_{\lambda\mu}T) = (f_\lambda m_{\lambda\mu}) T = f_\mu T .$$

By the property of limits there exists a unique homomorphism $\phi : AT \longrightarrow C$ such that $\phi\, p_\lambda = f_\lambda T$, for each $\lambda \in \Lambda$. Thus we get the following commutative diagram ,

$$
\begin{tikzcd}
C \arrow[r, "p_\lambda"] \arrow[rrr, bend left, "\sigma^\natural"] & C_\lambda T \arrow[r, "\theta_\lambda"] & C_\lambda & C/\sigma \arrow[l, dashed, "\phi_\lambda"'] \\
& AT \arrow[ul, dashed, "\phi"] \arrow[u, dashed, "f_\lambda T"] \arrow[r, "\theta"] & A \arrow[u, "f_\lambda"'] &
\end{tikzcd}
$$

where $\theta : AT \longrightarrow A$ is the reflection homomorphism for $A \in \mathcal{C}$.

Now let $\rho = \{(a, b) \in AT \times AT : a\theta = b\theta\}$. Then ρ is a congruence on AT which is the kernel of θ . Therefore, for each $(a, b) \in \rho$ and for each $\lambda \in \Lambda$, we have from the above diagram :

$$
\begin{aligned}
a\,\theta\,f_\lambda &= b\,\theta\,f_\lambda , \\
a\,(f_\lambda T)\,\theta_\lambda &= b\,(f_\lambda T)\theta_\lambda , \\
a\,\phi\,p_\lambda\,\theta_\lambda &= b\,\phi\,p_\lambda\,\theta_\lambda , \\
a\,\phi\,q_\lambda &= b\,\phi\,q_\lambda .
\end{aligned}
$$

This shows that $(a\phi, b\phi) \in \sigma$. So there exists a unique homomorphism $\psi : A \longrightarrow C/\sigma$ such that $\theta\psi = \phi\sigma^\natural$, since θ is a surjection. Now, for each $\lambda \in \Lambda$, again from the above diagram we get :

$$\theta\,\psi\,\phi_\lambda = \phi\,\sigma^\natural\,\phi_\lambda = \phi\,p_\lambda\,\theta_\lambda = (f_\lambda T)\theta_\lambda = \theta\,f_\lambda .$$

Again as θ is a surjection, we get from here $\psi\,\phi_\lambda = f_\lambda$, for each $\lambda \in \Lambda$.

Next we show that ψ is unique with this property. We first show that for $c, d \in C/\sigma$, $c\,\phi_\lambda = d\,\phi_\lambda$ for each $\lambda \in \Lambda$, implies that $c = d$. There exist $c', d' \in C$ such that $c'\sigma^{\natural} = c$ and $d'\sigma^{\natural} = d$. This implies that

$$(c'\sigma^{\natural}\phi_\lambda = d'\sigma^{\natural}\phi_\lambda, \text{ for each } \lambda \in \Lambda),$$

$$\Longrightarrow \quad (c'\,q_\lambda = d'\,q_\lambda, \text{ for each } \lambda \in \Lambda),$$

$$\Longrightarrow \quad ((c', d') \in \sigma),$$

$$\Longrightarrow \quad c = c'\sigma^{\natural} = d'\sigma^{\natural} = d. \qquad \text{(I)}$$

Now let $\psi' : A \longrightarrow C/\sigma$ be a homomorphism in $\mathscr{C}$ such that $\psi'\phi_\lambda = f_\lambda$ for each $\lambda \in \Lambda$. Then for all $a \in A$ we have

$$a\,\psi'\phi_\lambda = a\,f_\lambda = a\,\psi\,\phi_\lambda,$$

for each $\lambda \in \Lambda$. Therefore from (I) we get $a\,\psi' = a\,\psi$ for all $a \in A$. Hence $\psi' = \psi$. Thus we have proved the following theorem :

Theorem 3.1.1. Let $\mathscr{C}$ be an algebraic category with the following property : "For every family $\{\alpha_\lambda : A \longrightarrow A_\lambda\}_{\lambda \in \Lambda}$ of homomorphisms in $\mathscr{C}$, the homomorphism $\rho^{\natural} : A \longrightarrow A/\rho$ and the induced homomorphisms $\overline{\alpha}_\lambda : A/\rho \longrightarrow A\,\alpha_\lambda$, $\lambda \in \Lambda$, are in $\mathscr{C}$, where $\rho = \cap\{\rho_\lambda : \lambda \in \Lambda\}$ and ρ_λ is the kernel of α_λ for each $\lambda \in \Lambda$". Let $\mathscr{D}$ be a reflective subcategory of $\mathscr{C}$. such that each reflection homomorphism $\theta_C : CT \longrightarrow C$ is surjective, where $T : \mathscr{C} \longrightarrow \mathscr{D}$ is the reflector functor. If for a diagram D in $\mathscr{D}$ over a scheme (Λ, M, d) a limit exists in $\mathcal{D}$, then for same diagram in $\mathcal{C}$ a limit exists in $\mathcal{C}$.

We prove the dual of the above result in the following form, using definitions 1.1.12 and 1.1.13.

Theorem 3.1.2. Let $\mathcal{C}$ be an algebraic category with unions and surjective images. Let $\mathcal{D}$ be a coreflective subcategory of $\mathcal{C}$ such that each coreflection $\theta_C : C \longrightarrow CT$ is a monomorphism onto $\operatorname{Im} \theta_C$ and $\theta_C^{-1} : C\,\theta_C \longrightarrow C$ is in $\mathcal{C}$, where $T : \mathcal{C} \longrightarrow \mathcal{D}$ is the coreflector functor and $\theta_C\, \theta_C^{-1} = I_C$. If for a diagram D in $\mathcal{D}$ a colimit exists in $\mathcal{D}$, then for the same diagram in $\mathcal{C}$ a colimit exists in $\mathcal{C}$.

Proof. Let $\{ C_\lambda : \lambda \in \Lambda \}$ be the family of objects involved in the diagram D in $\mathcal{C}$. Then we have the corresponding diagram in $\mathcal{D}$ involving the family $\{ C_\lambda T : \lambda \in \Lambda \}$ of objects such that for each $m \in M$ the following diagram is commutative,

$$\begin{tikzcd} C_\lambda \arrow[r, "\theta_\lambda"] \arrow[d, "m_{\lambda\mu}"'] & C_\lambda T \arrow[d, dashed, "m_{\lambda\mu}T"] \\ C_\mu \arrow[r, "\theta_\mu"] & C_\mu T \end{tikzcd}$$

where $\theta_\lambda : C_\lambda \longrightarrow C_\lambda T$, for each $\lambda \in \Lambda$, is the coreflection in $\mathcal{D}$ of $C_\lambda \in \mathcal{C}$.

Let $\{ \alpha_\lambda : C_\lambda T \longrightarrow C \}_{\lambda \in \Lambda}$ be a colimit of D in $\mathcal{D}$. Then $C' = \cup \{ \operatorname{Im} \theta_\lambda\, \alpha_\lambda : \lambda \in \Lambda \}$ is a subobject of C . Now we have

$$C_\lambda \xrightarrow{\psi_\lambda} \operatorname{Im} \theta_\lambda \alpha_\lambda \xrightarrow{i_\lambda} C' \text{, for each } \lambda \in \Lambda .$$

Clearly $\psi_\lambda i_\lambda i_{C'} = \theta_\lambda \alpha_\lambda$, where $i_{C'} : C' \longrightarrow C$ is the inclusion morphism. We claim that $\{ \psi_\lambda i_\lambda : C_\lambda \longrightarrow C' \}_{\lambda \in \Lambda}$ is a colimit of D in $\mathcal{C}$. For each $m \in M$ we have from above:

$$m_{\lambda\mu}(\psi_\mu i_\mu i_{C'}) = m_{\lambda\mu} \theta_\mu \alpha_\mu = \theta_\lambda(m_{\lambda\mu} T) \alpha_\mu = \theta_\lambda \alpha_\lambda = \psi_\lambda i_\lambda i_{C'} .$$

Since $i_{C'}$ is a monomorphism we get

$$m_{\lambda\mu}(\psi_\mu i_\mu) = \psi_\lambda i_\lambda$$

for each $m \in M$. So $\{ \psi_\lambda i_\lambda : C_\lambda \longrightarrow C' \}_{\lambda \in \Lambda}$ is cocompatible for D. Now let $\{ f_\lambda : C_\lambda \longrightarrow B \}_{\lambda \in \Lambda}$ be a cocompatible family for D. Then, since $\{ f_\lambda T : C_\lambda T \longrightarrow BT \}_{\lambda \in \Lambda}$ is cocompatible for D in $\mathcal{D}$, we get a commutative diagram:

$$\begin{tikzcd}
 & \operatorname{Im} \theta_\lambda \alpha_\lambda \arrow[r, "i_\lambda"] & C' \arrow[d, "i_{C'}"] \\
C_\lambda \arrow[ur, "\psi_\lambda"] \arrow[r, "\theta_\lambda"] \arrow[d, "f_\lambda"'] & C_\lambda T \arrow[r, "\alpha_\lambda"] \arrow[d, dashed, "f_\lambda T"] & C \arrow[dl, dashed, "\phi"] \\
B \arrow[r, "\theta"] & BT &
\end{tikzcd}$$

for a uniquely defined morphism $\phi : C \longrightarrow BT$.
Let $\psi = \phi / C'$, i.e. $\psi = i_{C'} \phi$. Then for each $\lambda \in \Lambda$ we have:

$$\psi_\lambda i_\lambda \psi = \psi_\lambda i_\lambda i_{C'} \phi = f_\lambda \theta .$$

So $(\operatorname{Im} \theta_\lambda \alpha_\lambda) i_\lambda i_{C'} \phi \subseteq B\theta$, for each $\lambda \in \Lambda$, since images are surjective. By definition of C', $(C') i_{C'} \phi \subseteq B\theta$. Then

$\psi\,\theta^{-1}$ is defined and we get

$$(\psi_\lambda \iota_\lambda)(\psi\,\theta^{-1}) = \psi_\lambda \iota_\lambda(\iota_{C'}\,\phi\,\theta^{-1}) = \theta_\lambda \alpha_\lambda \phi\,\theta^{-1} = \theta_\lambda(f_\lambda T)\,\theta^{-1}$$

$$= f_\lambda \theta\,\theta^{-1} = f_\lambda\,,$$

for each $\lambda \in \Lambda$. Moreover it can be easily seen that $\psi\,\theta^{-1}$ is unique with this property. This completes the proof.

Now we give an abstract version of theorem 3.1.1.

Let $\mathcal{C}$ be a category with kernels, cokernels and intersection of kernels. Let $\mathcal{D}$ be a reflective subcategory of $\mathcal{C}$ such that each reflection morphism $\theta_C : CT \longrightarrow C$ is the cokernel of its kernel. Suppose that for a diagram D in $\mathcal{D}$ a limit exists in $\mathcal{D}$ and consider the same diagram in $\mathcal{C}$. Let $\{ C_\lambda : \lambda \in \Lambda \}$ be the family of objects of $\mathcal{C}$ involved in D. Then we get the corresponding diagram D in $\mathcal{D}$, involving the family $\{ C_\lambda T : \lambda \in \Lambda \}$ of objects of $\mathcal{D}$, such that for each $m \in M$ we have the following commutative diagram:

$$\begin{tikzcd} C_\lambda T \arrow[r, "\theta_\lambda"] \arrow[d, "m_{\lambda\mu} T"'] & C_\lambda \arrow[d, "m_{\lambda\mu}"] \\ C_\mu T \arrow[r, "\theta_\mu"] & C_\mu \end{tikzcd}$$

Let $\{ p_\lambda : C \longrightarrow C_\lambda T \}_{\lambda \in \Lambda}$ be a limit of D in $\mathcal{D}$. Let $u_\lambda : X_\lambda \longrightarrow C$ be the kernel of $q_\lambda = p_\lambda \theta_\lambda : C \longrightarrow C_\lambda$, for each $\lambda \in \Lambda$. Then $u : X = \cap \{ X_\lambda : \lambda \in \Lambda \} \longrightarrow C$ exists. Let $\pi : C \longrightarrow L$ be the cokernel of $u : X \longrightarrow C$. Now we have $u\, q_\lambda = 0$, for each $\lambda \in \Lambda$. Therefore by the definition of cokernels there exists a unique morphism $\phi_\lambda : L \longrightarrow C_\lambda$, for each $\lambda \in \Lambda$, such that $\pi\, \phi_\lambda = q_\lambda$ and we get a commutative diagram:

$$\begin{tikzcd}
X_\lambda \arrow[dr, "u_\lambda"] & & L \arrow[dd, dashed, "\phi_\lambda"] & u\,\pi = 0 \\
& C \arrow[ur, "\pi"] \arrow[dr, "q_\lambda"'] & & \\
X \arrow[uu, dashed, "y_\lambda"] \arrow[ur, "u"'] & & C_\lambda & u_\lambda q_\lambda = 0
\end{tikzcd}$$

where $y_\lambda : X \longrightarrow X_\lambda$ is a unique morphism such that $y_\lambda u_\lambda = u$. So we get a family $\{ \phi_\lambda : L \longrightarrow C_\lambda \}_{\lambda \in \Lambda}$ of morphisms in $\mathcal{C}$, which we now show to be compatible for D as follows:

$\pi\, \phi_\lambda\, m_{\lambda\mu} = q_\lambda\, m_{\lambda\mu} = p_\lambda\, \theta_\lambda\, m_{\lambda\mu} = p_\lambda (m_{\lambda\mu} T)\, \theta_\mu = p_\mu\, \theta_\mu = q_\mu = \pi\, \phi_\mu$.

But $\pi : C \longrightarrow L$ is an epimorphism, by 1.1.15. So we get:

$\phi_\lambda\, m_{\lambda\mu} = \phi_\mu$, for each $m \in M$.

Now let $\{ f_\lambda : B \longrightarrow C_\lambda \}_{\lambda \in \Lambda}$ be a compatible family for D in $\mathcal{C}$. Then we get a compatible family $\{ f_\lambda T : BT \longrightarrow C_\lambda T \}_{\lambda \in \Lambda}$ for D in $\mathcal{D}$. Let $\nu : K \longrightarrow BT$ be the kernel of $\theta : BT \longrightarrow B$. Then, for each $\lambda \in \Lambda$, we get the following commutative diagram:

$$\begin{tikzcd}
 & & & & L \arrow[dl, dashed, "\phi_\lambda"'] \\
X_\lambda \arrow[r, "u_\lambda"] & C \arrow[r, "p_\lambda"] \arrow[urrr, "\pi"] & C_\lambda T \arrow[r, "\theta_\lambda"] & C_\lambda & \\
X \arrow[u, dashed, "y_\lambda"] \arrow[ur, "u"'] & K \arrow[r, "\nu"] & BT \arrow[ul, dashed, "\phi"] \arrow[u, dashed, "f_\lambda T"'] \arrow[r, "\theta"] & B \arrow[u, "f_\lambda"'] &
\end{tikzcd}$$

Therefore for each $\lambda \in \Lambda$, we get:

$$\nu\, \phi\, q_\lambda = \nu\, \phi\, p_\lambda\, \theta_\lambda = \nu\, (f_\lambda T)\, \theta_\lambda = \nu\, \theta\, f_\lambda = 0\, f_\lambda = 0 .$$

Hence there exists a unique morphism $\alpha_\lambda : K \longrightarrow X_\lambda$, for each $\lambda \in \Lambda$, such that $\alpha_\lambda u_\lambda = \nu \phi$. Now by the definition of intersections, there exists a unique morphism $\alpha : K \longrightarrow X$ such that $\alpha\, y_\lambda = \alpha_\lambda$, for each $\lambda \in \Lambda$. Therefore we get:

$$\nu\, \phi\, \pi = \alpha_\lambda u_\lambda\, \pi = \alpha\, y_\lambda\, u_\lambda\, \pi = \alpha\, u\, \pi = \alpha\, 0 = 0 .$$

Since $\theta : BT \longrightarrow B$ is the cokernel of its kernel, there exists a unique morphism $\psi : B \longrightarrow L$ such that $\theta\, \psi = \phi\, \pi$. Hence, for each $\lambda \in \Lambda$, we get

$$\theta\, \psi\, \phi_\lambda = \phi\, \pi\, \phi_\lambda = \phi\, q_\lambda = \theta\, f_\lambda .$$

By 1.1.15, θ is an epimorphism. So we get $\psi\, \phi_\lambda = f_\lambda$, for each $\lambda \in \Lambda$. Clearly ψ is unique with this property.

Example. Let $\mathcal{C}$ be the category of all finitely generated abelian groups and $\mathcal{D}$ be the subcategory of $\mathcal{C}$, of all finitely generated torsion free abelian groups. Then $\mathcal{C}$ and $\mathcal{L}$ satisfy the hypotheses of the above abstract result.

We leave the statement of the dual of this abstract result, as one can state it in analogy with theorem 3.1.2.

Chapter 4

Some Functors

Throughout this chapter we will use the following notation.

$\mathcal{A}$ = the category of all d.g. near-rings.

$\mathcal{B}$ = the category of all faithful d.g. near-rings.

$\mathcal{S}$ = the category of all sets.

$\mathcal{S}$ = the category of all semigroups.

$\mathcal{G}$ = the category of all groups.

$\mathcal{G}_{(R,S)}$ = the category of all (R,S)-groups for a d.g. near-ring (R,S) .

4.1. Functor : $\mathcal{A} \times \mathcal{S} \longrightarrow \mathcal{G}$.

First of all we give a result which we are going to use in this section.

Theorem 4.1.1. [12] Let $Fr(X,R,S)$ be the free (R,S)-group on the set X . Let X be the disjoint union of the subsets $\{ X_\lambda : \lambda \in \Lambda \}$. Then $Fr(X,R,S) \cong \underset{\lambda \in \Lambda}{*} (R,S)\ Fr(X_\lambda,R,S)$, the free (R,S)-product of the family $\{ Fr(X_\lambda,R,S) : \lambda \in \Lambda \}$ of subgroups of $Fr(X,R,S)$.

Remark 4.1.2. If $(R,S) \in \mathcal{B}$, then by 4.1.1. we have

$$Fr(X,R,S) \cong \underset{x \in X}{*} (R,S)\ Fr(x,R,S) ,$$

$$\cong \underset{x \in X}{*} (R,S)\ \{ Gp\{x\} * R_x \} , \text{ by } 1.6.12.$$

Therefore $Fr(X,R,S)$ is a group generated by the set

$\{ x , r_x : x \in X , r \in (R,S) \}$.

Now we prove a few technical lemmas.

Lemma 4.1.3. Let $h : (R,S) \longrightarrow (T,U)$ be a homomorphism in $\mathfrak{B}$. Then there exists an (R,S)-homomorphism

$$\overline{h} : Fr(X,R,S) \longrightarrow Fr(X,T,U) ,$$

which maps r_x to $(rh)_x$ and x to x for all $x \in X$, $r \in (R,S)$, X being any set.

Proof. By lemma 2.1.2 $Fr(X,T,U)$ is an (R,S)-group. Since $Fr(X,R,S)$ is the free (R,S)-group on X , there exists a unique (R,S)-homomorphism $\overline{h} : Fr(X,R,S) \longrightarrow Fr(X,T,U)$ extending the identity map on X (1.6.6) . Clearly $x\overline{h} = x$, for all $x \in X$. Now $r_x \overline{h} = (xr)\overline{h} = (x\overline{h})(rh) = x(rh) = (rh)_x$.

Remark. The identity d.g. near-ring homomorphism $I_{(R,S)}$ gives us the identity (R,S)-homomorphism

$$I_{Fr(X,R,S)} : Fr(X,R,S) \longrightarrow Fr(X,R,S) ,$$

for every set X .

Lemma 4.1.4. Let $\alpha : X \longrightarrow Y$ be a mapping in $\mathcal{S}$, and let $(R,S) \in \mathfrak{B}$. Then there exists an (R,S)-homomorphism

$$\underline{\alpha} : Fr(X,R,S) \longrightarrow Fr(Y,R,S)$$

such that

$$\underline{\alpha} / X = \alpha \quad \text{and} \quad r_x \underline{\alpha} = r_{x\alpha} .$$

Proof. Let ι_Y be the inclusion map

$$Y \longrightarrow Fr(Y,R,S) .$$

Then $\alpha \iota_Y$ is a mapping from X to $Fr(Y,R,S)$. So there exists a unique (R,S)-homomorphism

$$\underline{\alpha} : Fr(X,R,S) \longrightarrow Fr(Y,R,S)$$

extending the map $\alpha \iota_Y$. Therefore $\underline{\alpha} / X = \alpha \iota_Y = \alpha$.

Now $(r_x)\,\underline{\alpha} = (x\,r)\,\underline{\alpha} = (x\,\underline{\alpha})\,r = (x\,\alpha)\,r = r_{x\,\alpha}$.

Remark. $I_X : X \longrightarrow X$ gives rise to the identity (R,S)-homomorphism $I_{Fr(X,R,S)} : Fr(X,R,S) \longrightarrow Fr(X,R,S)$.

Lemma 4.1.5. For $(R,S) \in \mathcal{A}$ and $X \in \mathcal{S}$, the groups $Fr(X,R,S)$ and $Fr(X,\underline{R},\underline{S})$ are isomorphic.

Proof. Let ϕ , μ be the representations of $Fr(X,R,S)$ and $Fr(X,\underline{R},\underline{S})$ as (R,S)- and $(\underline{R},\underline{S})$-groups respectively. Let θ be the natural homomorphism from (R,S) to $(\underline{R},\underline{S}) = (R,S)/A$, where A is as defined in lemma 2.1.3. Then by the same lemma , $\operatorname{Ker}\phi \supseteq A$. By lemma 2.1.2 , $Fr(X,\underline{R},\underline{S})$ is an (R,S)-group with representation $\theta\,\mu$. Since $Fr(X,R,S)$ is the free (R,S)-group on X , there exists a unique (R,S)-homomorphism

$$\psi : Fr(X,R,S) \longrightarrow Fr(X,\underline{R},\underline{S})$$

extending the identity map

$$\iota_X : X \subseteq Fr(X,R,S) \longrightarrow X \subseteq Fr(X,\underline{R},\underline{S}) .$$

Obviously ψ is an epimorphism.

Now $r \in \operatorname{Ker}\phi$ implies that $g(r\phi) = 0$ in $Fr(X,R,S)$ for each $g \in Fr(X,R,S)$, and therefore $g(r\phi)\psi = 0$ in $Fr(X,\underline{R},\underline{S})$. So $0 = g(r\phi)\psi = (g\psi)r\theta\mu = h(r\theta\mu)$ for all h in $Fr(X,\underline{R},\underline{S})$ since ψ is an epimorphism. Therefore $r\theta\mu = 0$ in $(E(H),End(H))$, where $H = Fr(X,\underline{R},\underline{S})$. Hence $r\theta = 0$ in $(\underline{R},\underline{S})$ as μ is a faithful representation. Therefore $r \in A$ and we get $\operatorname{Ker}\phi \subseteq A$. Hence we have $\operatorname{Ker}\phi = A$. So there exists a unique d.g. near-ring homomorphsim $\eta : (\underline{R},\underline{S}) \longrightarrow (E(G),End(G))$, where $G = Fr(X,R,S)$, such that $\theta\eta = \phi$. Thus $Fr(X,R,S)$ is an $(\underline{R},\underline{S})$-group with representation η. Again as $Fr(X,\underline{R},\underline{S})$ is the free $(\underline{R},\underline{S})$-group on X, there exists a unique $(\underline{R},\underline{S})$-homomorphism

$$\alpha : Fr(X,\underline{R},\underline{S}) \longrightarrow Fr(X,R,S)$$

extending the identity map of X.

Now for $x(r\phi) \in Fr(X,R,S)$ we have

$$\begin{aligned} \{x(r\phi)\}\psi\alpha &= \{(x\psi)r\theta\mu\}\alpha \\ &= \{x(r\theta\mu)\}\alpha \\ &= (x\alpha)r\theta\eta \\ &= x(r\theta\eta) \\ &= x(r\phi). \end{aligned}$$

Similarly we can show that $\{x(r\theta\mu)\}\alpha\psi = x(r\theta\mu)$.

Now the set $Y = \{ x , x(r\phi) : x \in X , r \in (R,S) \}$ generates the group $Fr(X,R,S)$ while the set

$$Z = \{ x , x(r\theta\mu) : x \in X , r \in (R,S) \}$$

generates the group $Fr(X,\underline{R},\underline{S})$. Also $\psi\alpha$ maps Y identically onto itself and $\alpha\psi$ maps Z identically onto itself. Therefore $\psi\alpha = I_{Fr(X,R,S)}$ and $\alpha\psi = I_{Fr(X,\underline{R},\underline{S})}$ and hence $Fr(X,R,S)$ is isomorphic to $Fr(X,\underline{R},\underline{S})$.

Now we are in a position to prove the following theorem :

Theorem 4.1.6 A function $P : \mathcal{B} \times \mathcal{S} \longrightarrow \mathcal{G}$, defined by $((R,S) , X) \longmapsto Fr(X,R,S)$ is a covariant functor such that $P / \mathcal{B} = P_X$ and $P / \mathcal{S} = P_{(R,S)}$ are covariant functors for each $X \in \mathcal{S}$ and each $(R,S) \in \mathcal{B}$ respectively. Also P is natural in both the components.

We divide this theorem into three parts and prove them as separate theorems.

Theorem 4.1.7 For each $X \in \mathcal{S}$, $P_X : \mathcal{B} \longrightarrow \mathcal{G}$, defined by $(R,S) \longmapsto Fr(X,R,S)$ is a covariant functor.

Theorem 4.1.8. For each $(R,S) \in \mathcal{B}$, $P_{(R,S)} : \mathcal{S} \longrightarrow \mathcal{G}$ defined by $X \longmapsto Fr(X,R,S)$ is a covariant functor.

Theorem 4.1.9. For each $\alpha : X \longrightarrow Y$ in $\mathcal{S}$ and each

$h : (R,S) \longrightarrow (T,U)$ in $\mathfrak{B}$, there are natural transformations

$$\alpha\,(R,S) : P_X \longrightarrow P_Y$$

$$h\,(X) : P_{(R,S)} \longrightarrow P_{(T,U)}$$

respectively.

Remark 4.1.10 Let H be an (R,S)-group for $(R,S) \in \mathfrak{B}$, with two (R,S)-homomorphisms α and β from $Fr(X,R,S)$ to H . In order to prove that $\alpha = \beta$, by 4.1.2 , it is sufficient to show that they agree on $\{ x , r_x : x \in X , r \in (R,S) \}$. This will be a common argument in the proofs of these theorems. We will use this without further mention.

Proof of theorem 4.1.7. We write $Fr(X,R,S) = (R,S)\,P_X$. Let $h : (R,S) \longrightarrow (T,U)$ be a homomorphism in $\mathfrak{B}$. By 4.1.3 we get an (R,S)-homomorphism $\overline{h} : Fr(X,R,S) \longrightarrow Fr(X,T,U)$ and we write $\overline{h} = h\,P_X$. Since $I_{(R,S)}$ in $\mathfrak{B}$ gives rise to $I_{Fr(X,R,S)}$ in $\mathcal{G}$, we get $I_{(R,S)}\,P_X = I_{Fr(X,R,S)}$.

Now let $h : (R,S) \longrightarrow (T,U)$ and $f : (T,U) \longrightarrow (Q,V)$ be homomorphisms in $\mathfrak{B}$. Then we have homomorphisms

$$h\,P_X : Fr(X,R,S) \longrightarrow Fr(X,T,U)$$

$$f\,P_X : Fr(X,T,U) \longrightarrow Fr(X,Q,V)$$

$$(h\,f)\,P_X : Fr(X,R,S) \longrightarrow Fr(X,Q,V)$$

in $\mathcal{G}$. To complete the proof we show that $(hP_X)(f\ P_X) = (h\ f)\ P_X$. For $x \in X$, $r \in (R,S)$, by 4.1.3 , we have :

$$x\ (h\ P_X)\ (f\ P_X) = x\ (f\ P_X) = x = x\ (h\ f)\ P_X$$

and

$$r_x(h\ P_X)\ (f\ P_X) = (r\ h)_x\ (f\ P_X) = (r\ hf)_x = r_x\ (h\ f)\ P_X\ .$$

Hence we get $(h\ P_X)\ (f\ P_X) = (hf)\ P_X$. This proves that P_X is a covariant functor.

Proof of theorem 4.1.8. We write $Fr(X,R,S) = X\ P_{(R,S)}$ and let $\alpha : X \longrightarrow Y$ be a map in $\mathcal{S}$. By 4.1.4 there exists an (R,S)-homomorphism $\underline{\alpha} : Fr(X,R,S) \longrightarrow Fr(Y,R,S)$. We denote $\underline{\alpha}$ by $\alpha\ P_{(R,S)}$. Since I_X gives rise to the identity (R,S)-homomorphism $I_{Fr(X,R,S)}$ in $\mathcal{G}$ we write $I_X\ P_{(R,S)} = I_{X\ P_{(R,S)}}$.

Now let $\alpha : X \longrightarrow Y$ and $\beta : Y \longrightarrow Z$ be maps in $\mathcal{S}$. By 4.1.4 we get (R,S)-homomorphisms :

$$\alpha\ P_{(R,S)} : Fr(X,R,S) \longrightarrow Fr(Y,R,S)$$

$$\beta\ P_{(R,S)} : Fr(Y,R,S) \longrightarrow Fr(Z,R,S)$$

$$(\alpha\ \beta)\ P_{(R,S)} : Fr(X,R,S) \longrightarrow Fr(Z,R,S)\ .$$

For $x \in X$ and $r \in (R,S)$, by 4.1.4 , we have :

$$x\ \{\alpha\ P_{(R,S)}\}\ \{\beta\ P_{(R,S)}\} = (x\ \alpha)\{\beta\ P_{(R,S)}\} = x(\alpha\ \beta) = x\{(\alpha\ \beta)\ P_{(R,S)}\},$$

$$r_x\{\alpha\ P_{(R,S)}\}\{\beta\ P_{(R,S)}\} = (r_{x\ \alpha})\{\beta\ P_{(R,S)}\} = r_{x(\alpha\ \beta)} = r_x\{(\alpha\ \beta)\ P_{(R,S)}\}\ .$$

Hence $\{\alpha\ P_{(R,S)}\}\{\beta\ P_{(R,S)}\} = \{(\alpha\beta)\ P_{(R,S)}\}$, which completes the proof.

Proof of theorem 4.1.9. Let $h : (R,S) \longrightarrow (T,U)$ be a homomorphism in $\mathcal{B}$ and $\alpha : X \longrightarrow Y$ be a map in $\mathcal{S}$. By theorems 4.1.3 , 4.1.4 , 4.1.7, 4.1.8 we get the following diagram in $\mathcal{G}$

$$\begin{tikzcd}
Fr(X,R,S) \arrow[r, "\alpha\ P_{(R,S)}"] \arrow[d, "h\ P_X"'] & Fr(Y,R,S) \arrow[d, "h\ P_Y"] \\
Fr(X,T,U) \arrow[r, "\alpha\ P_{(T,U)}"'] & Fr(Y,T,U)
\end{tikzcd}$$

where $Fr(X,R,S) = X\ P_{(R,S)} = (R,S)\ P_X$,

$Fr(X,T,U) = X\ P_{(T,U)} = (T,U)\ P_X$,

$Fr(Y,R,S) = Y\ P_{(R,S)} = (R,S)\ P_Y$,

$Fr(Y,T,U) = Y\ P_{(T,U)} = (T,U)\ P_Y$.

Now by the same theorems, we have ,for each $x \in X$ and $r \in (R,S)$:

$$\begin{aligned}
x\ \{h\ P_X\}\ \{\alpha\ P_{(T,U)}\} &= \{\ x(\ h\ P_X)\}\{\alpha\ P_{(T,U)}\} \\
&= x\{\ \alpha\ P_{(T,U)}\} \\
&= x\ \alpha\ ,
\end{aligned}$$

and $x\ \{\alpha\ P_{(R,S)}\}\{h\ P_Y\} = \{\ x(\alpha\ P_{(R,S)})\}\{\ h\ P_Y\}$

$$= (x\,\alpha)\{h\,P_Y\}$$

$$= x\,\alpha\ .$$

Thus $x\{h\,P_X\}\{\alpha\,P_{(T,U)}\} = x\{\alpha\,P_{(R,S)}\}\{h\,P_Y\}$,

and $r_x\{h\,P_X\}\{\alpha\,P_{(T,U)}\} = r_x\{\alpha\,P_{(R,S)}\}\{h\,P_Y\}$, for

$$\begin{aligned}
r_x\{h\,P_X\}\{\alpha\,P_{(T,U)}\} &= \{r_x(h\,P_X)\}\{\alpha\,P_{(T,U)}\} \\
&= \{(r\,h)_x\}\{\alpha\,P_{(T,U)}\} \\
&= (r\,h)_{x\,\alpha} \\
&= \{(r)_{x\,\alpha}\}\{h\,P_Y\} \\
&= \{r_x(\alpha\,P_{(R,S)})\}\{h\,P_Y\} \\
&= r_x\{\alpha\,P_{(R,S)}\}\{h\,P_Y\}\ .
\end{aligned}$$

Hence we get

$$\{h\,P_X\}\{\alpha\,P_{(T,U)}\} = \{\alpha\,P_{(R,S)}\}\{h\,P_Y\}\ ,$$

i.e. the above diagram is commutative. Hence h and α define the natural transformations:

$$h(X) : P_{(R,S)} \longrightarrow P_{(T,U)}\ ,$$

$$\alpha(R,S) : P_X \longrightarrow P_Y$$

respectively. This completes the proof of theorem 4.1.9.

Thus we have proved theorem 4.1.6.

Remark 4.1.11. $F \times I_{\mathcal{S}}$ and $G \times I_{\mathcal{S}} : \mathcal{A} \times \mathcal{S} \longrightarrow \mathcal{B} \times \mathcal{S}$, where F and G are the functors defined in chapter 2 , are covariant functors, which are coadjoint and adjoint respectively of the inclusion functor $I_{\mathcal{B}} \times I_{\mathcal{S}} : \mathcal{B} \times \mathcal{S} \longrightarrow \mathcal{A} \times \mathcal{S}$.

The following main theorem of this section becomes an obvious corollary of theorem 4.1.6 , by lemma 4.1.5.

Theorem 4.1.12. $C : \mathcal{A} \times \mathcal{S} \longrightarrow \mathcal{G}$ defined by $((R,S),X) \longmapsto Fr(X,R,S)$ is a covariant functor such that $C/\mathcal{A} = C_X$ and $C/\mathcal{S} = C_{(R,S)}$ are covariant functors for each $X \in \mathcal{S}$ and each $(R,S) \in \mathcal{A}$ respectively. Also C is natural in both components.

Note that $C_X = F\, P_X$ and $C_{(R,S)} = P_{(R,S)}\, F$. Also $P_{(R,S)}$ for each $(R,S) \in \mathcal{B}$ is a functor from the category $\mathcal{S}$ to the category $\mathcal{G}_{(R,S)}$, which is a subcategory of $\mathcal{G}$. Clearly $\mathcal{G}_{(R,S)}$ is not a full subcategory of $\mathcal{G}$.

The forgetful functor $Q_{(R,S)} : \mathcal{G}_{(R,S)} \longrightarrow \mathcal{S}$ is the functor which forgets the (R,S)-group structure on the objects of $\mathcal{G}_{(R,S)}$, i.e. if G is an (R,S)-group then $G\, Q_{(R,S)}$ is the underlying set $\underline{G}$ of G . If $\alpha : G \longrightarrow H$ is an (R,S)-homomorphism in $\mathcal{G}_{(R,S)}$ then $\alpha\, Q_{(R,S)} = \alpha$.

Theorem 4.1.13. The forgetful functor $Q_{(R,S)} : \mathcal{G}_{(R,S)} \longrightarrow \mathcal{S}$ is an adjoint of the functor $P_{(R,S)} : \mathcal{S} \longrightarrow \mathcal{G}_{(R,S)}$.

Proof. Let $\alpha : X \longrightarrow Y$ be a mapping in $\mathcal{S}$ and let $\beta : Fr(X,R,S) \longrightarrow G$, $\theta : G \longrightarrow H$ be homomorphisms in $\mathcal{G}_{(R,S)}$. Then we have a diagram :

$$\begin{tikzcd}
X \arrow[r, "\iota_X"] \arrow[d, "\alpha"'] & \mathrm{Fr}(X,R,S) = X\,P_{(R,S)} \arrow[d, dashed, "\alpha\,P_{(R,S)}"] \\
Y \arrow[r, "\iota_Y"] \arrow[d, dashed, "\beta\,\phi"'] & \mathrm{Fr}(Y,R,S) = Y\,P_{(R,S)} \arrow[d, "\beta"] \\
\underline{G} \arrow[r, "I_G"] \arrow[d, dashed, "\theta\,Q_{(R,S)}"'] & G \arrow[d, "\theta"] \\
\underline{H} \arrow[r, "I_H"] & H
\end{tikzcd}$$

where $\phi : \mathrm{Hom}_{\mathcal{G}_{(R,S)}}(Y\,P_{(R,S)}, G) \longrightarrow \mathrm{Hom}_{\mathcal{S}}(Y, G\,Q_{(R,S)})$, for each pair $(Y,G) \in \mathcal{S} \times \mathcal{G}_{(R,S)}$,
is defined by $\beta\phi = \beta / Y$. In this case it is rather trivial to show that ϕ is well defined and a bijection, therefore we are not going to prove **this.** But in the next section we will give the proof of a similar fact , since it is not so obvious in that case. It is easy to see that the above diagram is commutative. Therefore

$\{ (\alpha\,P_{(R,S)})\,\beta \}\,\phi = \alpha(\beta\phi)$ and $(\beta\,\theta)\,\phi = (\beta\,\phi)(\theta\,Q_{(R,S)})$.

Hence $Q_{(R,S)}$ is an adjoint of $P_{(R,S)}$.

Corollary 4.1.14. For each $(R,S) \in \mathcal{A}$, there exists a covariant functor $Q_{(R,S)\,F} : \mathcal{G}_{(R,S)\,F} \longrightarrow \mathcal{S}$, which is an adjoint of $P_{(R,S)F} : \mathcal{S} \longrightarrow \mathcal{G}_{J(R,S)F}$.

Proof. Obvious.

4.2. Functors : $\mathcal{A} \rightleftarrows \mathcal{S}$

We define $M : \mathcal{S} \longrightarrow \mathcal{A}$ by $S \longmapsto (Fr(S), S)$ for each $S \in \mathcal{S}$ and for each $\alpha : S \longrightarrow U$ in $\mathcal{S}$, αM to be the induced d.g. near-ring homomorphism $(Fr(S),S) \longrightarrow (Fr(U),U)$ in $\mathcal{A}$. Then $\alpha M / S = \alpha$. Now we prove the following theorem :

Theorem 4.2.1. $M : \mathcal{S} \longrightarrow \mathcal{A}$ is a covariant functor.

Proof. Clearly $I_S : S \longrightarrow S$ in $\mathcal{S}$ gives rise to $I_{(Fr(S),S)} : (Fr(S),S) \longrightarrow (Fr(S),S)$ in $\mathcal{A}$. Therefore $I_S M = I_{SM}$. Now let $f : S \longrightarrow U$ and $g : U \longrightarrow T$ be homomorphisms in $\mathcal{S}$. Then we get homomorphisms $f M$, $g M$ and $(f g) M$ in $\mathcal{A}$. To complete the proof we have to show that $(f M)(g M) = (f g) M$. For each $s \in S$ we have :

$s\{(f M)(g M)\} = \{s(f M)\}(g M) = (s f)(g M) = (s f) g = s(fg)$ and $s\{(f g) M\} = s(f g)$, i.e. $(f g) M / S = (f M)(g M) / S$. Hence by the property of free d.g. near-rings $(f g) M = (f M)(g M)$.

Now we define another functor $N : \mathcal{A} \longrightarrow \mathcal{S}$ by $(R,S) N = S$ for each $(R,S) \in \mathcal{A}$, and $\theta N = \theta / S$ for each homomorphism $\theta : (R,S) \longrightarrow (T,U)$ in $\mathcal{A}$. Then we prove the following theorem :

Theorem 4.2.2. $N : \mathcal{A} \longrightarrow \mathcal{S}$ is a covariant functor.

Proof. By definition of N, $I_{(R,S)} N = I_{(R,S)N}$.

Now let $f : (R,S) \longrightarrow (T,U)$ and $g : (T,U) \longrightarrow (Q,V)$ be homomorphisms in $\mathcal{A}$. Then for each $s \in S$, by definition of homomorphisms in $\mathcal{A}$, we get

$$s\{(f\ N)(g\ N)\} = \{s(fN)\}(g\ N) = (s\ f)(g\ N) = s(f\ g)$$
$$= s\{(f\ g)\ N\}.$$

Hence $(f\ N)(g\ N) = (f\ g)\ N$, which completes the proof.

Theorem 4.2.3. N is an adjoint of M.

Proof. To prove this theorem we show the existence of a bijection

$$\phi = \phi_{(U,(R,S))} : \mathrm{Hom}_{\mathcal{A}}(U\ M, (R,S)) \longrightarrow \mathrm{Hom}_{\mathcal{S}}(U, (R,S)\ N)$$

for each pair $(U, (R,S)) \in \mathcal{S} \times \mathcal{A}$, which is natural in both components. For that let $\alpha \in \mathrm{Hom}_{\mathcal{A}}(U\ M, (R,S))$ and define $\alpha\phi = \alpha / U$ in $\mathcal{S}$. It follows from the definition of homomorphisms in $\mathcal{A}$, that ϕ is well defined. Now let $\alpha, \beta \in \mathrm{Hom}_{\mathcal{A}}(U\ M, (R,S))$ such that $\alpha\phi = \beta\phi$, i.e. $\alpha / U = \beta / U$. Therefore by the property of free d.g. near-rings $\alpha = \beta$. So ϕ is an injection. Also for $\gamma \in \mathrm{Hom}_{\mathcal{S}}(U, (R,S)N)$, γM is a homomorphism from $(Fr(U),U)$ to $(Fr(S),S)$ in $\mathcal{A}$, such that

$\gamma M / U = \gamma$. Let $\pi : (Fr(S), S) \longrightarrow (R, S)$ be the natural homomorphism. Then $\pi / S = I_S$. Therefore $(\gamma M)\pi$ is a homomorphism from $(Fr(U), U)$ to (R, S) in $\mathcal{A}$. Now

$$\{(\gamma M)\pi\}\phi = \{(\gamma M)\pi\} / U \text{ in } \mathcal{S},$$

i.e. for each $u \in U$,

$$\begin{aligned} u\{[(\gamma M)\pi]\phi\} &= u\{(\gamma M)\pi\} \\ &= \{u(\gamma M)\}\pi \\ &= \{u\gamma\}\pi \\ &= u\gamma . \end{aligned}$$

So we get $\{(\gamma M)\pi\}\phi = \gamma$, which proves that ϕ is a surjection. Hence ϕ is a bijection.

Let $(U, (R,S))$ and $(T, (Q,V)) \in \mathcal{S} \times \mathcal{A}$, with homomorphisms $\alpha : U \longrightarrow T$ in $\mathcal{S}$ and $\theta : (R,S) \longrightarrow (Q,V)$, $\nu : TM \longrightarrow (R,S)$ in $\mathcal{A}$. Then we get a commutative diagram

$$\begin{tikzcd}
U \arrow[r, "i_U"] \arrow[d, "\alpha"'] & UM \arrow[d, dashed, "\alpha M"] \\
T \arrow[r, "i_T"] \arrow[d, dashed, "\nu\phi"'] & TM \arrow[d, "\nu"] \\
S = (R,S)N \arrow[r, "i_S"] \arrow[d, dashed, "\theta N"'] & (R,S) \arrow[d, "\theta"] \\
V = (Q,V)N \arrow[r, "i_V"] & (Q,V)
\end{tikzcd}$$

where ι_U, ι_T, ι_S, ι_V are inclusion semigroup homomorphisms.

Now $\{(\alpha M)\nu\}\phi = \{(\alpha M)\nu\} / U$, i.e. for each $u \in U$,

$$
\begin{aligned}
u[\{(\alpha M)\nu\}\phi] &= u\{(\alpha M)\nu\} \\
&= \{u(\alpha M)\}\nu \\
&= (u\alpha)\nu \\
&= (u\alpha)(\nu\phi) \\
&= u\{\alpha(\nu\phi)\}.
\end{aligned}
$$

Therefore $\{(\alpha M)\nu\}\phi = \alpha(\nu\phi)$.

We know that $(\nu\theta)\phi = (\nu\theta) / T$, i.e. for each $t \in T$,

$$
\begin{aligned}
t[(\nu\theta)\phi] = t(\nu\theta) = (t\nu)\theta &= \{t(\nu\phi)\}\theta \\
&= \{t(\nu\phi)\}(\theta N) \\
&= t\{(\nu\phi)(\theta N)\}.
\end{aligned}
$$

Therefore $(\nu\theta)\phi = (\nu\phi)(\theta N)$. This completes the proof that N is an adjoint of M.

Chapter 5

Inverse Semigroups of Endomorphisms

5.1. Preliminaries

A semigroup S is called regular if for each $x \in S$ there exists at least one element y in S such that $x y x = x$, and an element y is said to be an inverse of x if $x y x = x$, $y x y = y$. In a regular semigroup each element has an inverse which is not necessarily unique. An element $e \in S$ with $e^2 = e$ is called an idempotent. 0 and 1, if they are in S, are idempotents. In a regular semigroup S, for each $x \in S$, $x y$ and $y x$ are idempotents for every inverse y of x.

We call a semigroup S an inverse semigroup if each $x \in S$ possesses a unique inverse y in S and we denote this inverse by x^{-1}.

Theorem 5.1.1. A semigroup S is an inverse semigroup if and only if S is regular and the idempotents of S commute.

A lower semilattice is a partially ordered set in which each pair of elements has a greatest lower bound.

Theorem 5.1.2. Let $(E, \leqslant)$ be a lower semilattice. Then $(E, \wedge)$ is a commutative semigroup of idempotents and for all a, b in E, $a \leqslant b$ if and only if $a \wedge b = a$.

Let $(E, .)$ be a commutative semigroup of idempotents. Then the relation $\leqslant$ on E defined by, $a \leqslant b$ if and only if $a . b = a$, is a partial order on E such that $(E, \leq)$ is a lower semilattice. In $(E, \leq)$, $a \wedge b = a \cdot b$.

This result shows that the notions of commutative semigroup of idempotents and lower semilattice are equivalent. So the term semilattice can be used with either meaning.

Since the set E of idempotents of an inverse semigroup S forms a commutative semigroup, we call it the semilattice of idempotents of S.

Theorem 5.1.3. Let S be an inverse semigroup with semilattice E of idempotents. Then for all $x, y \in S$, $e \in E$,

1) $(x^{-1})^{-1} = x$,

2) $e^{-1} = e$,

3) $(x\,y)^{-1} = y^{-1}x^{-1}$,

4) $x\,e\,x^{-1}$ and $x^{-1}e\,x \in E$.

Corollary 5.1.4. If $x_1, ----, x_n$, n finite, are elements of an inverse semigroup S, then

$$(x_1 \cdot \ldots\ldots x_n)^{-1} = x_n^{-1} \cdot \ldots\ldots\ldots x_1^{-1}.$$

In particular $(x^n)^{-1} = (x^{-1})^n$ for each $x \in S$.

Theorem 5.1.5. Let S be an inverse semigroup with semilattice E of idempotents. Then

1) $S = \cup \{ eS : e \in E \}$,

2) for $e \in E$, $s \in S$, there exists an $f \in E$ such that $e s = s f$.

Theorem 5.1.6. Every homomorphic image of an inverse semigroup is an inverse semigroup.

Remarks 5.1.7.

1) Sub-semigroups of an inverse semigroup need not be inverse.

2) $s^{-1}\phi$ is the inverse in $S\phi$ of $s\phi$ for each $s \in S$, where S is an inverse semigroup and ϕ is a semigroup homomorphism.

Theorem 5.1.8. If ϕ is a semigroup homomorphism from an inverse semigroup S to a group G , then

1) $e\phi = 1_G$ for each $e \in E$,

2) $S\phi$ is a subgroup of G .

We define a relation $\leq$ on an inverse semigroup S by

" $x \leq y$ if and only if there exists $e \in E$ such that $x = e y$".

Then we have the following result :

Theorem 5.1.9. $(S, \leq)$ is a partially ordered set.

The restriction of $\leq$ to E is easily seen to be the natural semilattice ordering on E , " $e \leq f$ if and only if $e f = e$ ".

A semigroup S is called right reductive if for $a , b \in S$, $x a = x b$, for all $x \in S$ implies that $a = b$.

Theorem 5.1.10. An inverse semigroup is right reductive.

A congruence on a semigroup S is an equivalence relation ρ

on the set S such that

$$(x, y), (a, b) \in \rho \text{ implies that } (xa, yb) \in \rho$$

for all $x, y, a, b \in S$.

A congruence ρ on a semigroup S is called a group congruence if S/ρ is a group.

Theorem 5.1.11. Let S be an inverse semigroup with semilattice E of idempotents. Then the relation

$$\sigma = \{ (x, y) \in S \times S : ex = ey \text{ for some } e \in E \}$$

is the minimum group congruence on S.

Definition 5.1.12. Let S be a semigroup. S is called a semilattice of groups if there exists a semilattice Y such that

1) $\{ S_\alpha : \alpha \in Y \}$ is a family of disjoint subgroups of S indexed by Y;

2) for each pair $\alpha, \beta \in Y$ with $\alpha \geqslant \beta$ there exists a homomorphism $\phi_{\alpha,\beta} : S_\alpha \longrightarrow S_\beta$ such that

a) $\phi_{\alpha,\alpha}$ is the identity automorphism of S_α for each $\alpha \in Y$;

b) $\phi_{\alpha,\beta}\phi_{\beta,\gamma} = \phi_{\alpha,\gamma}$ for every $\alpha, \beta, \gamma \in Y$ such that $\alpha \geqslant \beta \geqslant \gamma$.

Definition 5.1.13. A semigroup S is called a Clifford semigroup if it is regular and its idempotents are central, i.e., $es = se$ for all $s \in S$, and for each idempotent $e \in S$.

Theorem 5.1.14.

1) A semigroup S is a semilattice of groups if and only if it is a Clifford semigroup.

2) An inverse semigroup S is a semilattice of groups if and only if $ss^{-1} = s^{-1}s$ for all $s \in S$.

Definition 5.1.15. Let G be a group. G is called a semidirect product or split extension of its subgroups H and K if

1) K is normal in G,

2) $H \cap K = \{0\}$,

3) $G = K + H$.

Theorem 5.1.16 If a group G is a semidirect product of its subgroup H and a normal subgroup K then there is a natural projection $\pi : G \longrightarrow H$ with $\operatorname{Ker} \pi = K$.

Theorem 5.1.17. [4] Let S be an inverse semigroup with semilattice E of idempotents and G a group such that $S \subseteq \operatorname{End}(G)$. Then for each $e \in E$, $G = \operatorname{Ker} e + \operatorname{Image} e$ is the semidirect product of its subgroups Image e and Ker e. Moreover if $e \leq f$ then

$$\operatorname{Ker} e \supseteq \operatorname{Ker} f \quad \text{and} \quad \operatorname{Image} e \subseteq \operatorname{Image} f,$$

and if $e < f$ then $\operatorname{Image} e \subset \operatorname{Image} f$, $\operatorname{Ker} e \supset \operatorname{Ker} f$.

Inverse semigroups arise naturally as sets of partial $1-1$ mappings of a set. A partial $1-1$ mapping of a set is a mapping whose domain is a subset, possibly empty, of X. The set $\mathcal{I}(X)$ of partial $1-1$ mappings of X is a subset of $B(X)$, the set of binary relations on X. So the elements of $\mathcal{I}(X)$ are multiplied by the law of composition of $B(X)$. We write $\operatorname{dom}\alpha$ for domain of α and $\operatorname{ran}\alpha$ for range of α, $\alpha \in \mathcal{I}(X)$. If $\alpha, \beta \in \mathcal{I}(X)$ then $(x, y) \in \alpha \circ \beta$ if and only if there exists $z \in X$ such that $(x, z) \in \alpha$ and $(z, y) \in \beta$, i.e. if and only if there exists $z \in \operatorname{ran}\alpha \cap \operatorname{dom}\beta$ such that $x\alpha = z$ and $z\beta = y$, i.e., if and only if $y = (x\alpha)\beta$, where $x \in (\operatorname{ran}\alpha \cap \operatorname{dom}\beta)\alpha^{-1}$. Thus $\alpha \circ \beta$, usually written as $\alpha\beta$, is a partial $1-1$ mapping with

$$\operatorname{dom}\alpha\beta = (\operatorname{ran}\alpha \cap \operatorname{dom}\beta)\alpha^{-1} \text{ and}$$

$$\operatorname{ran}\alpha\beta = (\operatorname{ran}\alpha \cap \operatorname{dom}\beta)\beta .$$

One can prove the following result.

Theorem 5.1.18.

1) $\mathcal{I}(X)$ is an inverse semigroup.

2) $\alpha \in \mathcal{I}(X)$ is an idempotent if and only if $\alpha = I_A$ for some subset A of X.

3) The semilattice of idempotents of $\mathcal{I}(X)$ is isomorphic to the semilattice of subsets of X under intersection.

Theorem 5.1.19. (Vagner–Preston Representation Theorem)

If S is an inverse semigroup then there exists a set X and a monomorphism $\phi : S \longrightarrow \mathcal{J}(X)$.

Definition 5.1.20. $\mathcal{J}(X)$ is called the symmetric inverse semigroup on X .

Definition 5.1.21. Let G be a group. A partial $1-1$ mapping of G , whose domain is a subgroup of G , and which is a homomorphim, is called a partial isomorphism.

Remark 5.1.22. The set of partial isomorphisms of a group G forms a sub-semigroup $\mathcal{J}_G$ of $\mathcal{J}(G)$, the symmetric inverse semigroup on G . $\mathcal{J}_G$ is an inverse subsemigroup of $\mathcal{J}(G)$.

Let S be any semigroup with identity. If for some a, b in S , $ab = 1$ but $ba \neq 1$ then a, b generate an inverse sub-semigroup $\mathcal{C} = \mathcal{C}(a, b)$ of S . Each element of $\mathcal{C}$ is uniquely expressed as $b^m a^n$, for $m, n \geq 0$. $\{ b^n a^n : n \geq 0 \}$ is the set of idempotents of $\mathcal{C}$.

Definition 5.1.23. $\mathcal{C}$ is called a bicyclic inverse semigroup.

Definition 5.1.24. A semilattice $(Y, \leq)$ is called updirected if for each pair x, y of elements of Y there exists a $z \in Y$ such that $x \leq z$ and $y \leq z$.

Example 5.2.1. Let $\{e, f, 0\}$ be a semigroup such that each element of S is idempotent and products of distinct elements are 0. Let (R,S) be a d.g. near-ring with 0 as the zero of $(R,+)$ taken to be non-abelian.
Then we have $e+f \in (R,S)$ such that

$$(e+f)e = e^2 + fe = e + 0 = e,$$

$$(e+f)f = ef + f^2 = 0 + f = f,$$

i.e., $e+f$ is a left identity for e and f and hence for (R,S). Therefore by 1.6.9, (R,S) has a faithful representation on $(R,+)$. So $S \subseteq \mathrm{End}(R,+)$. Let $K = \cap\{\mathrm{Ker}\, s : s \in S\}$. Then K is not trivial, since $e+f-e-f \in K$. Therefore $K \supseteq \delta(R,+)$, the derived group of $(R,+)$. So $(R/K,+)$ is abelian. In this case K is an ideal of (R,S), by 1.5.9, for

1) $RK \subseteq K$ since $(rk)s = r(ks) = r0 = 0$, for all $s \in S$, $k \in K$;

2) $KS \subseteq K$, since $ks = 0 \in K$ for $k \in K$ and all $s \in S$.

Now $(R,S)/K$ is a d.g. near-ring with $(R/K,+)$ abelian. Therefore by 1.5.6.(2) $(R,S)/K$ is a ring.

Now we give an example of a d.g. near-ring over an inverse semigroup which does not have a left identity.

Example 5.2.2. Let E be an infinite, updirected semilattice of idempotents, which does not have a maximum element. Let

$(Fr(E),E)$ be the free d.g. near-ring on E. For an element $r = \epsilon_1 e_1 + ---- + \epsilon_n e_n$ of $(Fr(E),E)$, using induction on n, we can find an element $e \in E$ such that $e_i < e$ for $i = 1,---, n$.

Then
$$\begin{aligned} re &= (\epsilon_1 e_1 + ---- + \epsilon_n e_n)\, e \,, \\ &= \epsilon_1 e_1 e + --- + \epsilon_n e_n e \,, \\ &= \epsilon_1 e_1 + ---- + \epsilon_n e_n \,, \\ &= r \,, \\ &\neq e \,. \end{aligned}$$

This proves that no element of $(Fr(E),E)$ can be a left identity.

Example 5.2.3. Let F be a free group on

$$X = \{ x_1 , x_2 , ----, x_n , -- \} .$$

Define $\alpha : X \longrightarrow X$ by $x_{2n-1} \longmapsto x_{2n}$, $x_{2n} \longmapsto x_{2n+3}$, and $\beta : X \longrightarrow X$ by $x_3 , x_1 \longmapsto x_1$ and $x_{2n} \longmapsto x_{2n-1}$,

$$x_{2n+3} \longmapsto x_{2n}, \quad n \geq 1.$$

Then α is a $1-1$ mapping but not onto, while β is onto but not $1-1$. By the property of free groups α and β can be extended to endomorphisms of F, again denoted by α and β respectively. Also for $n \geq 1$, we have

$$x_{2n} \xmapsto{\alpha} x_{2n+3} \xmapsto{\beta} x_{2n}$$

$$x_{2n-1} \xmapsto{\alpha} x_{2n} \xmapsto{\beta} x_{2n-1} \,.$$

Therefore $\alpha\beta = I_X$ and hence $\alpha\beta = I_F$. But since

$$x_3 \xmapsto{\beta} x_1 \xmapsto{\alpha} x_2 \quad \text{and} \quad x_1 \xmapsto{\beta} x_1 \xmapsto{\alpha} x_2$$

we get $\beta\alpha \neq I_X$ and therefore $\beta\alpha \neq I_F$. Hence the sub-semigroup $\mathcal{C} = \mathcal{C}(\alpha, \beta)$ of the semigroup of endomorphisms of F is a bicyclic inverse semigroup. Each element of $\mathcal{C}$ is uniquely expressible as $\beta^m \alpha^n$ for $m, n \geq 0$. The inverse of $\beta^m \alpha^n$ is $\beta^n \alpha^m$ and $\{ \beta^n \alpha^n : n \geq 0 \}$ is the set of idempotents of $\mathcal{C}$ such that

$$1 > \beta\alpha > \beta^2\alpha^2 > ---- > \beta^n\alpha^n > -- > ---- \quad (5.2.3(a))$$

By 5.1.17 we get the following infinite chains of subgroups of F :

$$0 \subset \operatorname{Ker} \beta\alpha \subset \operatorname{Ker} \beta^2\alpha^2 \subset ---- \subset \operatorname{Ker} \beta^n\alpha^n \subset -- \quad (5.2.3(b))$$

$$F \supset \operatorname{Im} \beta\alpha \supset \operatorname{Im} \beta^2\alpha^2 \supset ------ \supset \operatorname{Im} \beta^n\alpha^n \supset -- \quad (5.2.3(c))$$

Now $F\beta^n\alpha^n \subseteq F\alpha^n = F(\alpha^n\beta^n)\alpha^n$, as $\alpha^n\beta^n = I_F$

$$= F\alpha^n(\beta^n\alpha^n) \subseteq F\beta^n\alpha^n ,$$

i.e. $F\beta^n\alpha^n = F\alpha^n$. $\quad$ (5.2.3(d))

Also $0 = g\beta^n\alpha^n$ implies that $0 = g\beta^n\alpha^n\beta^n = g\beta^n$, i.e.

$$\operatorname{Ker} \beta^n\alpha^n \subseteq \operatorname{Ker} \beta^n ,$$

and $0 = g\beta^n$ implies that $0 = g\beta^n\alpha^n$, i.e.

$$\operatorname{Ker} \beta^n \subseteq \operatorname{Ker} \beta^n\alpha^n .$$

Hence we get $\operatorname{Ker} \beta^n\alpha^n = \operatorname{Ker} \beta^n$. $\quad$ (5.2.3(e))

Therefore the infinite series (5.2.3(b)) and (5.2.3(c)) can be written as

$$0 \subset \operatorname{Ker} \beta \subset \operatorname{Ker} \beta^2 \subset ------ \subset \operatorname{Ker} \beta^n \subset --- (5.2.3(f))$$

$$F \supset \operatorname{Im} \alpha \supset \operatorname{Im} \alpha^2 \supset ------- \supset \operatorname{Im} \alpha^n \supset --- (5.2.3(g)).$$

Since F is free, by the definition of β, an element g of F belongs to $\operatorname{Ker} \beta$ only if there are cancellations in $g\beta$, and there can be cancellations in $g\beta$ if and only if $g\beta$ has pairs like

$$\epsilon_i x_i \beta + \epsilon_{i+1} x_{i+1} \beta = \pm x_1 \mp x_1 .$$

This can happen if and only if $\epsilon_i x_i + \epsilon_{i+1} x_{i+1}$ is $\pm x_3 \mp x_1$, or $\pm x_2 \mp x_1$ or $\pm x_3 \mp x_2$, or the inverses of such elements.

Case I $g\beta$ is exhausted by the cancellations of these types.

Case II After such cancellations we are left with elements of the type $h\beta - h\beta$.

In either case we have $g \in \{Gp(\pm x_3 \mp x_1, \pm x_2 \mp x_1)\}^F$.

Hence we have proved that

$$\operatorname{Ker} \beta = \{Gp(\pm x_3 \mp x_1, \pm x_2 \mp x_1)\}^F$$

Similarly we can show that

$$\operatorname{Ker} \beta^2 = \{Gp(\pm x_5 \mp x_1, \pm x_4 \mp x_1, \pm x_3 \mp x_1, \pm x_2 \mp x_1)\}^F$$

Now suppose that $\operatorname{Ker} \beta^m = \{Gp(\pm x_{2i+1} \mp x_1, \pm x_{2i} \mp x_1 : i \leqslant m)\}^F$, for $m \leqslant n$, and consider $\operatorname{Ker} \beta^{m+1}$;

$$g \in \operatorname{Ker} \beta^{m+1} \Longleftrightarrow g\beta \in \operatorname{Ker} \beta^m ,$$

$$\Longleftrightarrow \quad g\beta \in [\, Gp\{ \pm x_{2i+1} \mp x_1 ,\ \pm x_{2i} \mp x_1 : i \leq m \}\,]^F ,$$

by the induction hypothesis.

$$\Longleftrightarrow \quad g \in [\, Gp\{ \pm x_{2i+1} \mp x_1 ,\ \pm x_{2i} \mp x_1 : i \leq m+1 \}\,]^F ,$$

as before.

Hence $\operatorname{Ker} \beta^{m+1} = [\, Gp\{ \pm x_{2i+1} \mp x_1 ,\ \pm x_{2i} \mp x_1 : i \leq m+1 \}\,]^F$.
This completes the induction argument and it is proved that for all finite $n > 0$,

$$\operatorname{Ker} \beta^{n} = [Gp\{ \pm x_{2i+1} \mp x_1 ,\ \pm x_{2i} \mp x_1 : i \leq n \}]^F .$$

Next we find the subgroups $\{ \operatorname{Im} \alpha^n : n > 0 \}$. From the definition of α we see that $x_1 , x_3 \notin \operatorname{Im} \alpha$, whereas the other x_i's are in $\operatorname{Im} \alpha$. Therefore we get:

$$\operatorname{Im} \alpha = Gp\{ X \setminus \{ x_1 , x_3 \} \} .$$

Similarly we can show the following:

$$\operatorname{Im} \alpha^2 = Gp\{ X \setminus \{ x_1 , x_2 , x_3 , x_4 \} \} ,$$

$$\operatorname{Im} \alpha^3 = Gp\{ X \setminus \{ x_1 , x_2 , x_3 , x_4 , x_5 , x_7 \} \} ,$$

$$\operatorname{Im} \alpha^4 = Gp\{ X \setminus \{ x_1 , x_2 , x_3 , x_4 , x_5 , x_6 , x_7 , x_8 \}\} .$$

Now suppose that for each $m \leq n$, m even, we have;

$$\operatorname{Im} \alpha^{m-1} = Gp\{ X \setminus \{ x_i : i \leq 2(m-1)-1 ,\ x_{2(m-1)+1} \}\} ,$$

$$\operatorname{Im} \alpha^{m} = Gp\{ X \setminus \{ x_i : i \leq 2m \}\} .$$

We know that, $x_{2i+3} = x_{2i}\alpha$ and $x_{2i} = x_{2i-1}\alpha$. Since $\{x_{2i}, x_{2i-1} : i \leq m\} \not\in \operatorname{Im} \alpha^{m}$, we get $\{x_{2i+3}, x_{2i} : i \leq m\} \not\in \operatorname{Im} \alpha^{m+1}$. Therefore $\{x_{i+3}, x_{i} : i \leq 2m\} \not\in \operatorname{Im} \alpha^{m+1}$. Also $x_{2m+1} = x_{2(m-1)}\alpha$ and since $x_{2(m-1)} \not\in \operatorname{Im} \alpha^{m}$, $x_{2m+1} \not\in \operatorname{Im} \alpha^{m+1}$. It is easy to see that the other x_i's belong to $\operatorname{Im} \alpha^{m+1}$. Hence

$$\operatorname{Im} \alpha^{m+1} = \operatorname{Gp}(X \mid \{x_i : i \leq 2m+1, x_{2m+3}\}) . \qquad \text{(III)}$$

Again from the definition of α and $\operatorname{Im} \alpha^{m+1}$ we get that

$$\{x_i : i \leq 2(m+2)\} \not\in \operatorname{Im} \alpha^{m+2},$$

whereas the other x_i's are in $\operatorname{Im} \alpha^{m+2}$. Thus we get

$$\operatorname{Im} \alpha^{m+2} = \operatorname{Gp}(X \mid \{x_i : i \leq 2(m+2)\} . \qquad \text{(IV)}$$

Hence we have proved that (I) and (II) imply (III) and (IV). This completes the induction argument and we get for all $n > 0$,

$$\operatorname{Im} \alpha^{n} = \operatorname{Gp}(X \mid \{x_i : i \leq 2n\}) , \text{ if } n \text{ is even,}$$

$$= \operatorname{Gp}(X \mid \{x_i : i \leq 2n-1, x_{2n+1}\}) , \text{ if } n \text{ is}$$

odd.

Example 5.2.4. Let (R,S) be a d.g. near-ring. Since S is a distributive semigroup, the elements of S are endomorphisms of $(R,+)$. But two distinct elements of S may define the same endomorphism of $(R,+)$.

Now let (R,S) be a d.g. near-ring with S an inverse semigroup. By 5.1.10, for $s \neq t$ in S, there exists $x \in S$ such that $x s \neq x t$. Therefore s and t define two distinct endomorphisms of $(R,+)$. Hence in this case we get $S \subseteq \mathrm{End}\,(R,+)$.

From now onwards, in this chapter, unless otherwise stated, S will denote an inverse semigroup with semilattice E of idempotents.

5.3. Representations of Inverse Semigroups

Throughout this section G will be an additive group such that $S \subseteq \mathrm{End}\,(G)$. Therefore for each $s \in S$, Gs, $G s^{-1}$ are subgroups of G. Also

$$G s s^{-1} \subseteq G s^{-1} = G s^{-1} s s^{-1} \subseteq G s s^{-1} ,$$

$$G s^{-1} s \subseteq G s = G s s^{-1} s \subseteq G s^{-1} s ,$$

i.e. $G s = G s^{-1} s$ and $G s^{-1} = G s s^{-1}$.

We define $\bar{s} = s / G s^{-1}$. Then for all $g \in G$ we have

$$g s = g(s s^{-1} s) = (g s s^{-1}) \bar{s} ,$$

and if $g s^{-1} \neq 0$, then we get

$$0 \neq g\,s^{-1} = g\,(s^{-1}s\,s^{-1}) = \{(g\,s^{-1})\,\overline{s}\}\,s^{-1}.$$

Since s^{-1} is an endomorphism of G, this implies that

$$(g\,s^{-1})\,\overline{s} \neq 0 \;.$$

Therefore $\overline{s}$ is a $1-1$ mapping from $G\,s^{-1}$ onto $G\,s$. Also, being a restriction of a homomorphism, $\overline{s}$ is a homomorphism. Hence each $\overline{s}$ is a partial isomorphism of G. We write $G\,s^{-1} = \operatorname{dom}\overline{s}$ and $G\,s = \operatorname{ran}\overline{s}$. Now we prove the following result :

Theorem 5.3.1. Let $S \subseteq \operatorname{End}(G)$. Then $\overline{S} = \{\overline{s} : s \in S\}$ is a semigroup of partial isomorphisms of G, which is isomorphic to S.

Proof. We define $\psi : S \longrightarrow \overline{S}$ by $s\,\psi = \overline{s}$. Then ψ is well defined and onto. For $s, t \in S$, $s\,\psi = t\,\psi$ implies that $\operatorname{dom}\overline{s} = \operatorname{dom}\overline{t}$ and for all x in this common domain $x\,\overline{s} = x\,\overline{t}$. We know that for all $g \in G$,

$$g\,s = g\,(ss^{-1}s) = (g\,s\,s^{-1})\,\overline{s} = (g\,s\,s^{-1})\,\overline{t} = g\,s\,s^{-1}t \;.$$

Therefore $s = s\,s^{-1}t$, i.e. $s \leq t$. Similarly we can show that $t \leq s$. Hence we get $s = t$. Thus ψ is $1-1$.

Finally we prove that ψ is a semigroup homomorphism. Since $G\,t^{-1}s^{-1} \subseteq G\,s^{-1}$, we have

$$\operatorname{dom}\overline{s\,t} \subseteq \operatorname{dom}\overline{s}$$

for all $s, t \in S$. Now for all $g \in G$, as S is inverse, we have

$$(g\,t^{-1}s^{-1})\,\overline{s} = g\,t^{-1}s^{-1}s = g\,t^{-1}t\,t^{-1}s^{-1}s = g\,t^{-1}s^{-1}s\,t\,t^{-1}.$$

Since right hand side is in $\operatorname{dom}\overline{t}$ we get $(\operatorname{dom}\overline{s\,t})\,\overline{s} \subseteq \operatorname{dom}\overline{t}$, and hence $\mathrm{dom}\ \overline{s\,t} \subseteq \mathrm{dom}\ \bar{s}\,\bar{t}$. Conversely, as

$$\mathrm{dom}\ \bar{s}\,\bar{t} = \{ x \in \mathrm{dom}\ \bar{s} \ : \ x\,\bar{s} \in \mathrm{dom}\ \bar{t} \} ,$$

for $x \in \mathrm{dom}\ \bar{s}\,\bar{t}$, $x = g\,s^{-1}$ for some $g \in G$ such that $x\,\bar{s} = (g\,s^{-1})\,s = h\,t^{-1}$ for some $h \in G$. Therefore

$$x = g\,s^{-1} = g\,s^{-1}s\,s^{-1} = (g\,s^{-1}s)\,s^{-1} = (h\,t^{-1})\,s^{-1}$$

$= h\,(t^{-1}s^{-1}) \in \mathrm{dom}\ \overline{s\,t}$. This shows that $\mathrm{dom}\ \bar{s}\,\bar{t} \subseteq \mathrm{dom}\ \overline{s\,t}$. Hence we get $\mathrm{dom}\ \bar{s}\,\bar{t} = \mathrm{dom}\ \overline{s\,t}$.

For all $g \in G$,

$$(g\,t^{-1}s^{-1})\,\bar{s}\,\bar{t} = (g\,t^{-1}s^{-1}s)\,\bar{t} = g\,t^{-1}s^{-1}s\,t = (g\,t^{-1}s^{-1})\,\overline{s\,t} ,$$

i.e. $x\,\bar{s}\,\bar{t} = x\,\overline{s\,t}$ for all $x \in \mathrm{dom}\ \bar{s}\,\bar{t} = \mathrm{dom}\ \overline{s\,t}$. So $\bar{s}\,\bar{t} = \overline{s\,t}$ or $(s\,\psi)(t\psi) = (s\,t)\,\psi$ for all $s, t \in S$. Hence ψ is a semigroup homomorphism. This completes the proof.

Note that $\bar{s}\,\bar{t}$ and $\overline{s\,t}$ are partial isomorphisms of G with same domain and with same action on this common domain. Therefore $\bar{s}\,\bar{t}$ and $\overline{s\,t}$ have the same range.

For $e \in E$, $\mathrm{dom}\ e = G\,e^{-1} = G\,e = \mathrm{ran}\ e$, and $\bar{e}$ is the identity automorphism of $G\,e$.

Corollary 5.3.2. The semilattice $(E, .)$ is isomorphic to the semilattice $\{ (\mathrm{dom}\ \bar{e}, \cap) \ : \ e \in E \}$ of subgroups of G.

Proof. From 5.3.1 and 5.1.1 we have $\mathrm{dom}\ \bar{e}\,\bar{f} = \mathrm{dom}\ \overline{e\,f} = \mathrm{dom}\ \overline{f\,e} = \mathrm{dom}\ \bar{f}\,\bar{e}$, and again by 5.3.1

$$\mathrm{dom}\ \bar{e}\,\bar{f} \subseteq \mathrm{dom}\ \bar{e} \cap \mathrm{dom}\ \bar{f} .$$

Conversely, if $x \in \operatorname{dom} \bar{e} \cap \operatorname{dom} \bar{f}$, then

$$x = g e = h f \text{ , for some } g , h \in G \text{ and}$$

$$x = h f = h(f f) = (h f) f = (g e) f = g (e f) \in \operatorname{dom} \overline{e f} .$$

Therefore $\operatorname{dom} \bar{e} \cap \operatorname{dom} \bar{f} \subseteq \operatorname{dom} \overline{e f}$. Hence we get

$$\operatorname{dom} \bar{e} \cap \operatorname{dom} \bar{f} = \operatorname{dom} \overline{e f} \qquad \text{(I)} .$$

Also if $\operatorname{dom} \bar{e} = \operatorname{dom} \bar{f}$, then for each $g \in G$, $g e = h f$ for some $h \in G$, and $g(e f) = h(f f) = h f = g e$. This gives us $e f = e$, i.e. $e \leq f$. Similarly we can show that $f \leq e$, so that $e = f$. Therefore

$$\operatorname{dom} \bar{e} = \operatorname{dom} \bar{f} \Longrightarrow e = f \quad \forall e , f \in E \qquad \text{(II)} .$$

Now we define $\phi : e \longrightarrow \operatorname{dom} \bar{e}$, then (I) and (II) show that ϕ is an isomorphism of $(E , .)$ onto $\{(\operatorname{dom} \bar{e} , \cap) : e \in E \}$. This completes the proof.

Theorem 5.3.3. For each $s \in S$, there is a split extension $G = \operatorname{Ker} s + \operatorname{dom} \bar{s}$ such that for all $s , t \in S$, we have

1) $\operatorname{Ker} s \subseteq \operatorname{Ker} s t$,
2) $(\operatorname{Ker} s t \cap \operatorname{dom} \bar{s}) \bar{s} \subseteq \operatorname{Ker} t$.

Proof. For $s \in S$, if $g \in (\operatorname{Ker} s \cap \operatorname{dom} \bar{s})$, then

$$g = h s^{-1} = h (s^{-1} s s^{-1}) = g (s s^{-1}) = 0 s^{-1} = 0 .$$

Therefore $\operatorname{Ker} s \cap \operatorname{dom} \bar{s} = \{0\}$. Moreover, for each $g \in G$,

$$(g - g s s^{-1}) s = g s - g s s^{-1} s = g s - g s = 0 .$$

So $g - g s s^{-1} = h \in \operatorname{Ker} s$, and since $g s s^{-1} \in \operatorname{dom} \bar{s}$,

$g = h + g s s^{-1} \in (\text{Ker } s + \text{dom } \bar{s})$, i.e. $G \subseteq \text{Ker } s + \text{dom } \bar{s}$.

Hence $G = \text{Ker } s + \text{dom } \bar{s}$ is a split extension.

For each $s \in S$, $g \in \text{Ker } s \Longrightarrow g(s\,t) = 0\,t = 0 \;\; \forall s \in S$,

$$\Longrightarrow g \in \text{Ker } s\,t \,.$$

Therefore $\text{Ker } s \subseteq \text{Ker } s\,t$, for all $s, t \in S$. This completes the proof of (1).

For $s, t \in S$,

$g \in (\text{Ker } s\,t \cap \text{dom } \bar{s}) \Longrightarrow g\,s\,t = 0$ and $g = h\,s^{-1}$,

for some $h \in G$.

And $g\,\bar{s}\,t = (h\,s^{-1})\,\bar{s}\,t = (h\,s^{-1}s)\,t = g\,s\,t = 0$.

Therefore $g\,\bar{s} \in \text{Ker } t$ and hence we get

$$(\text{Ker } s\,t \cap \text{dom } \bar{s})\,\bar{s} \subseteq \text{Ker } t \,.$$

This proves (2).

Corollary 5.3.4. For each $s \in S$, $\text{Ker } s = \text{Ker } e$ for some idempotent $e \in E$.

Proof. For each $s \in S$, we have from above

$$\text{Ker } s \subseteq \text{Ker } s\,s^{-1} \subseteq \text{Ker } (s\,s^{-1}s) = \text{Ker } s \,.$$

Therefore $\text{Ker } s = \text{Ker } s\,s^{-1}$, where $s\,s^{-1}$ is an idempotent.

Remark 5.3.5. Corollary 5.3.4 is true even if S is a regular semigroup with $S \subseteq \text{End}(G)$.

Corollary 5.3.6. If $s\,s^{-1} \leqslant s^{-1}s$ ($s^{-1}s \leqslant s\,s^{-1}$), $s \in S$, then $\text{Ker } s \supseteq \text{Ker } s^{-1}$ ($\text{Ker } s^{-1} \supseteq \text{Ker } s$).

Proof. $s s^{-1} \leq s^{-1}s \Longrightarrow \mathrm{Ker}\ s s^{-1} \supseteq \mathrm{Ker}\ s^{-1}s$ (by 5.1.17).

Therefore, by 5.3.4, we get

$$\mathrm{Ker}\ s = \mathrm{Ker}\ s s^{-1} \supseteq \mathrm{Ker}\ s^{-1}s = \mathrm{Ker}\ s^{-1} .$$

Corollary 5.3.7. If S is a semilattice of groups, then $\mathrm{Ker}\ s = \mathrm{Ker}\ s^{-1}$, for each $s \in S$.

Proof. By 5.1.14.(2) we have in this case $s s^{-1} = s^{-1}s$, for each $s \in S$. Therefore by 5.3.4,

$$\mathrm{Ker}\ s = \mathrm{Ker}\ s s^{-1} = \mathrm{Ker}\ s^{-1}s = \mathrm{Ker}\ s^{-1} .$$

Let T be an inverse semigroup of partial isomorphisms of a group G. For each $t \in T$, we denote by D_t and R_t the domain and range of t respectively. Obviously $D_{tu} \subseteq D_t$ and $R_{tu} \subseteq R_u$ for all $t, u \in T$. Now we prove the converse of 5.3.3 combined with 5.3.1.

Theorem 5.3.8. Let T be an inverse semigroup of partial isomorphisms of a group G such that for each $t \in T$ there exists a normal subgroup K_t of G satisfying

1) $G = K_t + D_t$ is a split extension,

$$\left.\begin{array}{ll} 2) & K_t \subseteq K_{tu} \\ 3) & (K_{tu} \cap D_t)\, t \subseteq K_u \end{array}\right] \text{ for all } t, u \in T .$$

Then T can be embedded in $\mathrm{End}\,(G)$.

Proof. For each $t \in T$, by (1), there exists a natural projection $\pi_t : G \longrightarrow D_t$. Then $\pi_t t$ is an endomorphism of G.

We define $\alpha : T \longrightarrow \mathrm{End}\,(G)$ by $t\,\alpha = \pi_t t$. Clearly α is well defined. For $t, u \in T$, by (1) each $g \in G$ is expressed as $g = x + y$, where $x \in K_{tu}$, $y \in D_{tu}$. So

$$g\,(t\,u)\,\alpha = g(\pi_{tu} t\,u) = (x + y)\,\pi_{tu} t\,u = y(t\,u). \qquad (\mathrm{I}).$$

Now $x = x_1 + y_1$, where $x_1 \in K_t$ and $y_1 \in D_t$. Therefore

$$y_1 = -x_1 + x \in (K_t + K_{tu}) \subseteq K_{tu} \qquad (\text{by } (2)).$$

So $y_1 \in (K_{tu} \cap D_t)$ and hence by (3) $y_1 t \in K_u$. Also $(y_1 + y) \in (D_t + D_{tu}) \subseteq D_t$. Therefore

$$g = x + y = x_1 + (y_1 + y),$$

where right hand side belongs to $(K_t + D_t)$.

$$\begin{aligned}
\text{Now} \quad g\,(t\,\alpha)(u\,\alpha) &= \{g\,(t\,\alpha)\}(u\,\alpha) \\
&= [\{x_1 + (y_1 + y)\}\,\pi_t t\,]\,(\pi_u u) \\
&= [(y_1 + y)\,t](\pi_u u) \\
&= (y_1 t + y\,t)(\pi_u u), \text{ as } y_1 \text{ and } y \in D_t, \\
&= (y\,t)u, \text{ as } y\,t \in D_u,\ y_1 t \in K_u, \\
&= y\,(t\,u). \qquad (\mathrm{II}).
\end{aligned}$$

Now (I) and (II) together imply that

$$g\{(t\,u)\,\alpha\} = g\,\{(t\,\alpha)(u\,\alpha)\}.$$

Since this is true for all $g \in G$, we get

$$(t\,u)\,\alpha = (t\,\alpha)(u\,\alpha).$$

Hence α is a semigroup homomorphism.

To complete the proof we need to show that α is 1—1.

For that let $t\alpha = u\alpha$ for some $t, u \in T$. Then

$$\pi_t t = \pi_u u .$$

Now since T is an inverse semigroup

$$t^{-1}\alpha = (t\alpha)^{-1} = (u\alpha)^{-1} = u^{-1}\alpha .$$

Therefore $\pi_{t^{-1}} t^{-1} = \pi_{u^{-1}} u^{-1}$, so $R_{t^{-1}} = R_{u^{-1}}$. But since $R_{t^{-1}} = D_t$, $R_{u^{-1}} = D_u$, we get $D_t = D_u$. Also for all $x \in D_t = D_u$,

$$x t = x(\pi_t t) = x(\pi_u u) = x u .$$

Therefore $t = u$ and hence α is an injection.

Remark. It is easy to check that $(T\alpha)\psi = T$, where ψ is the mapping defined in the proof of theorem 5.3.1.

Theorem 5.3.9. Let $S \subseteq \mathrm{End}\,(G)$, and let $K = \bigcap_{s \in S} \mathrm{Ker}\, s$. Then S is embedded in $\mathrm{End}\,(G/K)$ and the corresponding K is trivial.

Proof. For each $s \in S$ we define $s\alpha : G/K \longrightarrow G/K$ by $g + K \longmapsto gs + K$. Then for $g, h \in G$,

$$\begin{aligned} g + K = h + K &\Longrightarrow (g - h) \in K \\ &\Longrightarrow (g - h)s = 0 \\ &\Longrightarrow gs = hs \\ &\Longrightarrow gs + K = hs + K, \\ &\Longrightarrow (g + K)(s\alpha) = (h + K)(s\alpha) . \end{aligned}$$

Therefore $s\alpha$ is well defined. Now

$$\{(g+K)+(h+K)\}(s\alpha) = \{(g+h)+K\}(s\alpha)$$
$$= (g+h)s+K$$
$$= (gs+hs)+K$$
$$= (gs+hs)+K$$
$$= (gs+K)+(hs+K)$$
$$= (g+K)(s\alpha)+(h+K)(s\alpha).$$

Thus $s\alpha$, for each $s \in S$, is an endomorphism of G/K and we get a mapping $\alpha : s \longmapsto s\alpha$, from S to End (G/K). We now prove that α is a semigroup homomorphism which is $1-1$. Let $s, t \in S$. Then for all $g+K \in G/K$ we have

$$(g+K)\{(st)\alpha\} = \{g(st)\}+K$$
$$= \{(gs)t+K\}$$
$$= \{gs+K\}(t\alpha)$$
$$= (g+K)(s\alpha)(t\alpha).$$

Therefore $(st)\alpha = (s\alpha)(t\alpha)$, i.e. α is a semigroup homomorphism.

Now let $s\alpha = t\alpha$, for some $s, t \in S$. Then for all $g \in G$ we have $(g+K)(s\alpha) = (g+K)(t\alpha)$. Therefore for all $g \in G$, $gs+K = gt+K$. This implies that $(gs-gt)$ is in K. Therefore $0 = (gs-gt)(s^{-1}s)$

$$= g(ss^{-1}s) - g(ts^{-1}s)$$

$$= g s - g t s^{-1} s \ .$$

This implies that $g s = g t s^{-1} s$, for all $g \in G$. Therefore $s = t s^{-1} s$ and hence $s \leq t$. Similarly we can show that $t \leq s$. Thus we get $s = t$. This proves that α is $1-1$, and therefore $S \subseteq \text{End}\,(G/K)$.

Now let $\overline{K} = \cap \{ \text{Ker}\ s\alpha : s \in S \} \subseteq G/K$.

$$
\begin{aligned}
g + K \in \overline{K} &\Longrightarrow \text{for all } s \in S,\ (g+K)(s\alpha) = \overline{0}, \\
&\Longrightarrow gs + K = \overline{0}, \\
&\Longrightarrow gs \in K, \\
&\Longrightarrow (gs)\, s^{-1}s = 0, \\
&\Longrightarrow gs = 0, \\
&\Longrightarrow g \in K, \\
&\Longrightarrow g + K = \overline{0}\ .
\end{aligned}
$$

Hence $\cap \{ \text{Ker}\ s\alpha : s \in S \} = \{\overline{0}\}$ in G/K .

5.4. D.G. Near-rings over Inverse Semigroups

J.D.P. Meldrum conjectured that "a d.g. near-ring (R,S) over an inverse semigroup S , is faithful with a faithful representation on $(R,+)$". Example 5.2.1 gave us a clue to prove this conjecture, which apparently looked difficult.

First we prove a few technical lemmas.

Lemma 5.4.1. Let (R,S) be a d.g. near-ring. Then each element $e \in E$, commutes with each element of the subgroup $Gp\{E\}$ of $(R, +)$.

Proof. Let $e \in E$ and $r \in Gp\{E\}$. Then

$$r = \epsilon_1 e_1 + \cdots + \epsilon_n e_n ,$$

where $e_i \in E$, $i = 1, \ldots, n$. Now we have

$$\begin{aligned} e\, r &= e(\epsilon_1 e_1 + \cdots + \epsilon_n e_n) \\ &= \epsilon_1 e\, e_1 + \cdots + \epsilon_n e\, e_n \\ &= \epsilon_1 e_1 e + \cdots + \epsilon_n e_n e \\ &= (\epsilon_1 e_1 + \cdots + \epsilon_n e_n) e \\ &= r\, e . \end{aligned}$$

From example 5.2.2 we see that it is not necessary for a d.g. near-ring (R,S) to contain a left identity. In the following lemma we prove that each sub near-ring of (R,S), generated by a finite subset of E, contains a left identity. We call these left identities "local left identities".

Lemma 5.4.2. Let (R,S) be a d.g. near-ring. For every finite subset $\{e_1, \ldots, e_n\}$ of E, there exists an element x in N.R. $\{e_1, \ldots, e_n\}$, the near-ring generated by this subset, such that

1) $x e_i = e_i$ for $i = 1, \ldots, n$

2) $x^2 = x$

3) $x e = e x$, for all $e \in E$.

Proof. We use induction to prove this result. It is trivially true for all subsets of E which contain only one element. We consider a subset of E, containing two elements only, say $\{ e_1 , e_2 \}$. Let $x = e_1 - e_1e_2 + e_2$. Then

$$
\begin{aligned}
x e_1 &= (e_1 - e_1e_2 + e_2)e_1 \\
&= e_1^2 - e_1e_2e_1 + e_2e_1 \\
&= e_1 - e_2e_1^2 + e_2e_1 \\
&= e_1 - e_2e_1 + e_2e_1 \\
&= e_1 .
\end{aligned}
$$

Similarly we can show that $x e_2 = e_2$. This proves (1) and it follows from here that x is a left identity for N.R.$\{ e_1 , e_2 \}$, and x belongs to this near-ring. Therefore $x^2 = x$. This proves (2).

Since $x \in \mathrm{Gp}\{ E \}$, it follows from lemma 5.4.1 that $x e = e x$ for all $e \in E$. This completes the proof of the lemma for subsets of E, which contain two elements.

Now we suppose that the lemma is true for all subsets of E which contain $n-1$ elements, for $n > 2$, and consider a subset $\{ e_1 , ----, e_n \}$ of E, containing n elements. By the induction hypothesis, there exists an element $x \in$ N.R.$\{ e_1 , ---, e_{n-1}\}$ satisfying the three conditions of the lemma. Let

$$y = x - e_nx + e_n .$$

Then for $i = 1, \ldots, n-1$, by 5.4.1 and induction hypothesis, we have

$$\begin{aligned} y e_i &= (x - e_n x + e_n) e_i \\ &= x e_i - e_n x e_i + e_n e_i \\ &= e_i - e_n e_i + e_n e_i \\ &= e_i , \end{aligned}$$

and

$$\begin{aligned} y e_n &= (x - e_n x + e_n) e_n \\ &= x e_n - e_n x + e_n \\ &= e_n . \end{aligned}$$

Therefore $y e_i = e_i$ for $i = 1, \ldots, n$. Hence y is a left identity for N.R.$\{e_1, \ldots, e_n\}$. This gives us $y^2 = y$. Since $y \in Gp\{E\}$, by lemma 5.4.1, $e y = y e$, for all $e \in E$. This completes the proof.

Corollary 5.4.3. Let S' be a finite inverse subsemigroup of S. Then the sub d.g. near-ring (R',S') of (R,S) has a left identity.

Proof. Since S' is finite, the number of idempotents in S' is also finite. Let $\{e_1, \ldots, e_n\}$ be the set of idempotents of S'. By lemma 5.4.2 there exists a left identity x in the near-ring generated by $\{e_1, \ldots, e_n\}$. We claim that x is a left identity of (R',S'). For that let $r \in (R',S')$. Then

$$r = \epsilon_1 t_1 + \cdots + \epsilon_k t_k , \quad t_i \in S' \quad i = 1, \ldots, k .$$

Therefore $r = \epsilon_1 f_1 t_1 + \cdots + \epsilon_k f_k t_k$, where $f_i = t_i t_i^{-1}$, for each i. Since each $f_i \in S'$, $\{f_1, \ldots, f_k\}$ is a subset of $\{ e_1 , \dots , e_n \}$. Therefore $x f_i = f_i$ for each i, and

we get
$$\begin{aligned} x r &= x(\epsilon_1 f_1 t_1 + \dots + \epsilon_k f_k t_k) \\ &= \epsilon_1 x f_1 t_1 + \dots + \epsilon_k x f_k t_k \\ &= \epsilon_1 f_1 t_1 + \dots + \epsilon_k f_k t_k \\ &= r . \end{aligned}$$

Now we come to the main result of this section.

Theorem 5.4.4. If S is an inverse semigroup, then every d.g. near-ring (R,S) is faithful with a faithful representation on $(R,+)$.

Proof. Obviously (R,S) has a right regular representation on $(R,+)$. Let $0 \neq (\epsilon_1 e_1 s_1 + \dots + \epsilon_n e_n s_n) = r \in (R,S)$. By lemma 5.4.2, a left identity x exists in N.R.$\{ e_1 , \dots , e_n \}$. As in the proof of 5.4.3 we can show that $x r = r$, which proves that for each non zero $r \in (R,S)$ there exists an element $x = x_r$, in $(R,+)$, such that $x r = r$, Hence by 1.6.11, (R,S) has a faithful representation on $(R,+)$.

Corollary 5.4.5. Let $K = \cap \{ \text{Ker } s : s \in S \}$ and let E_R be the set of idempotents of $(R,\cdot)$. Then $K \cap E_R = \{0\}$.

Proof. Let $r \in K$ then $r = \epsilon_1 s_1 + \dots + \epsilon_n s_n$, and
$$\begin{aligned} r^2 &= r(\epsilon_1 s_1 + \dots + \epsilon_n s_n) \\ &= \epsilon_1 r s_1 + \dots + \epsilon_n r s_n \\ &= 0 + \dots + 0 = 0 . \end{aligned}$$

Thus $r \in E_R$ only if $r = 0$. Hence we have proved the result.

Corollary 5.4.6. Let (R,S) be a d.g. near-ring. Then for every ideal I of (R,S), $(R,S)/I$ is faithful.

Proof. Since S is inverse, $(S+I)/I$ is an inverse semigroup by 5.1.6. Therefore, by 5.4.4, $(R,S)/I$ acts faithfully on $(R/I, +)$.

Theorem 5.4.7. Let (R,S) be a d.g. near-ring and let

$$I = \{ x \in (R,S) : x e = 0 \text{ for some } e \in E \} .$$

Then I is an ideal of (R,S).

Proof. Clearly $I = \cup \{ \operatorname{Ker} e : e \in E \}$ and $\forall\, e,f \in E$, $\operatorname{Ker} e \cup \operatorname{Ker} f \subseteq \operatorname{Ker} ef$. Therefore I is a normal subgroup of $(R, +)$.

Let $r \in (R,S)$, $s \in S$, and $x \in I$. Then there exists an e in E such that $x e = 0$. Now

$$(r x) e = r(x e) = r\, 0 = 0$$

implies that $(r x) \in I$, i.e. $R I \subseteq I$. Also, by 5.1.5 (2),

$$0 = (xe)s = x(e s) = x(sf) = (xs)f \text{ , for some } f$$

It folows that $(xs) \in I$, i.e. $I S \subseteq I$. Hence the result follows by 1.5.9.

Note that in general I is not a zero ideal, since in example 5.2.1 there are $e \in E$ with $\operatorname{Ker} e \neq \{0\}$.

Let $\theta : (R,S) \longrightarrow (R,S)/I$ be the natural homomorphism, where I is the ideal of (R,S) defined in 5.4.7. Then we prove the following result.

Theorem 5.4.8. $S\theta$ is a group.

Proof. $[\theta / S] = \{(s,t) \in S \times S : s\theta = t\theta\}$ is a congruence on S. Also

$$s\theta = t\theta \iff s + I = t + I \text{ in } (R,S)/I$$
$$\iff (s - t) \in I$$
$$\iff (s - t)e = 0 \text{ for some } e \in E$$
$$\iff se = te .$$

Hence $[\theta/S] = \{(s,t) \in S \times S : se = te \text{ for some } e \in E\}$, which is the minimum group congruence σ on S. Thus

$$S\theta = (S + I)/I = S/[\theta/S] = S/\sigma ,$$

is a group, and $\theta/S = \sigma^{\natural}$.

Let $\mathcal{C}$ be the category of all d.g. near-rings over inverse semigroups and let $\mathcal{D}$ be a subcategory of $\mathcal{C}$ which is the category of all d.g. near-rings over groups. Clearly $(R,S)/I$ defined above is in $\mathcal{D}$.

Theorem 5.4.9. $(R,S)/I$ together with the natural homomorphism $\theta : (R,S) \longrightarrow (R,S)/I$ is the coreflection of $(R,S) \in \mathcal{C}$ in $\mathcal{D}$.

Proof. Let ϕ be a d.g. near-ring homomorphism from (R,S) to $(T,G) \in \mathcal{D}$. Then, by 5.1.8 (2), $S\phi$ is a subgroup of G.

Let $r \in I$, and let $e \in E$ be such that $re = 0$. Then

$0 = 0\phi = (re)\phi = (r\phi)(e\phi) = r\phi$, (by 5.1.8 (1)).

Therefore $r \in \operatorname{Ker}\phi$. So there exists a unique d.g. near-ring homomorphism $\psi : (R,S)/I \longrightarrow (T,G)$ such that $\theta\psi = \phi$. This completes the proof.

Since each d.g. near-ring $(R,S) \in \mathcal{C}$, has an associated d.g. near-ring in $\mathcal{D}$, we get the following corollary.

Corollary 5.4.10. $\mathcal{D}$ is a coreflective subcategory of $\mathcal{C}$.

Suppose that the semilattice E of idempotents of S is updirected. Therefore, since $e \leq e'$ in E implies that $eS \subseteq e'S$ and conversely, for each pair eS, fS of sub-semigroups of S we get a sub-semigroup gS of S, for some $g \in E$, such that $eS \subseteq gS$, $fS \subseteq gS$. Consider the diagram D in $\mathcal{S}$ over the scheme (E,M,d), where for $e \leq f$, $md = (e,f)$, $m_{ef} : eS \longrightarrow fS$ is the inclusion map. Let $\{\theta_e : eS \longrightarrow S\}_{e \in E}$ be the family of inclusion homomorphisms. Then we prove the following lemma.

Lemma 5.4.11. $\{\theta_e : eS \longrightarrow S\}_{e \in E}$ is a colimit of D in $\mathcal{S}$.

Proof. For all $m \in M$ and all $s \in S$ we have

$$(eS)\, m_{ef}\theta_f = (eS)\,\theta_f = (eS)\,\theta_e ,$$

since $eS \subseteq fS$ and $\theta_f/(eS) = \theta_e$. Therefore $m_{ef}\theta_f = \theta_e$ for all $m \in M$, i.e. $\{\theta_e : eS \longrightarrow S\}_{e \in E}$ is a cocompatible family for D .

Let $\{ \alpha_e : eS \longrightarrow T \}_{e \in E}$ be a cocompatible family for D in $\mathcal{S}$. Define $\phi : S \longrightarrow T$ by $s\phi = s\alpha_{ss^{-1}}$. Note that $s = ss^{-1}s \in (ss^{-1})S$. Moreover $s\phi = s\alpha_e$ for each e in E such that $s \in eS$, for then $(ss^{-1})S \subseteq eS$. For s, t in S, $s \in eS$, $t \in fS$, there is a sub-semigroup gS for some $g \in E$, such that $eS \subseteq gS$, $fS \subseteq gS$. Then $st \in gS$. Now $(s\phi)(t\phi) = (s\alpha_g)(t\alpha_g) = (st)\alpha_g = (st)\phi$. Hence ϕ is a semigroup homomorphism. For each $s \in S$, $e \in E$, we have

$$(es)\theta_e\phi = (es)\phi = (es)\alpha_e .$$

Therefore $\theta_e\phi = \alpha_e$ for each $e \in E$. Moreover it is clear that ϕ is unique with this property. Thus we have proved the lemma.

Now $\{ (eR, eS) : e \in E \}$ is a family of sub d.g. near-rings of (R,S) each having a left identity, namely the corresponding e . $\{ \theta_e : (eR, eS) \longrightarrow (R,S) \}_{e \in E}$ is a family of d.g. near-ring inclusion homomorphisms. Moreover $e \leq f$ implies that $eR \subseteq fR$. So we get the corresponding diagram D over (E,M,d) in the category $\mathcal{A}$ of all d.g. near-rings. In fact D is in $\mathcal{B}$ since each eS has a left identity namely e . Note that $\mathcal{B}$ is the category of all faithful d.g. near-rings.

Theorem 5.4.12. $\{ \theta_e : (eR, eS) \longrightarrow (R,S) \}_{e \in E}$ is a colimit of D in $\mathcal{A}$.

Proof. Obviously the given family of homomorphisms is cocompatible for D. Let $\{ \alpha_e : (eR, eS) \longrightarrow (T,U) \}_{e \in E}$ be a cocompatible family for D in $\mathcal{A}$. Then, since $\{ \alpha_e : eS \longrightarrow U \}_e$ is cocompatible for D in $\mathcal{S}$, by lemma 5.4.11, there exists a unique semigroup homomorphism $\phi : S \longrightarrow U$ such that $\theta_e \phi = \alpha_e$ for each $e \in E$. To complete the proof we just need to show that ϕ extends to a group homomorphism, then the result will follow from 1.5.13 and from the uniqueness of ϕ with $\theta_e \phi = \alpha_e$, $e \in E$, as semigroup homomorphisms. Let $\epsilon_1 s_1 + \cdots + \epsilon_n s_n = 0$ in (R,S). Then

$$0 = \epsilon_1 s_1 + \cdots + \epsilon_n s_n = \epsilon_1 e_1 s_1 + \cdots + \epsilon_n e_n s_n .$$

As in example 5.2.2, there exists $g \in E$ such that $e_i < g$ and hence $e_i S \subseteq gS$ for $i = 1, \ldots, n$. Therefore

$$(\epsilon_1 e_1 s_1 + \cdots + \epsilon_n e_n s_n) \in (gR, gS),$$

and hence is zero in (gR, gS). Now

$$\begin{aligned}
\epsilon_1 s_1 \phi + \cdots + \epsilon_n s_n \phi &= \epsilon_1 e_1 s_1 \alpha_{e_1} + \cdots + \epsilon_n e_n s_n \alpha_{e_n} \\
&= \epsilon_1 e_1 s_1 \alpha_g + \cdots + \epsilon_n e_n s_n \alpha_g \\
&= (\epsilon_1 e_1 s_1 + \cdots + \epsilon_n e_n s_n) \alpha_g \\
&= 0 \alpha_g \\
&= 0,
\end{aligned}$$

as α_g is a d.g. near-ring homomorphism. Therefore ϕ is well-defined and hence can be extended to a group homomorphism.

Remark. Since each member of the family $\{(eR, eS) : e \in E\}$ of d.g. near-rings, is in $\mathfrak{B}$, the diagram D, defined above, is in $\mathfrak{B}$. We have seen, in chapter 2, that all colimits of diagrams in $\mathfrak{B}$ exist in $\mathfrak{B}$. Hence (R,S) is in $\mathfrak{B}$. Thus theorem 5.4.12 provides an elementary proof of faithfulness of a d.g. near-ring (R,S) over an inverse semigroup S with updirected semilattice E of idempotents. Note from example 5.2.1 that E need not be updirected in general.

Chapter 6

Group d.g. near-rings

6.1. Preliminaries

Let (R,S) be a faithful d.g. near-ring and let G be a multiplicative group. Let X be any set and define:

$$Y = X \times G = \{(x,g) : x \in X, g \in G\},$$

$$X_g = \{(x,g) : x \in X\}.$$

Let $F = Fr(Y,R,S)$ be the free (R,S)-group on the set Y and let $F_g = Fr(X_g,R,S)$ be the free (R,S)-group on the set X_g, for each $g \in G$. Then we can consider each F_g as a subgroup of F. Moreover we have $F = \underset{g \in G}{*}(R,S)\, F_g$, the free (R,S)-product of $\{F_g : g \in G\}$. Since for all $g, h \in G$, the free (R,S)-generating sets X_g, X_h of F_g and F_h respectively, are in $1-1$ correspondence, F_g and F_h are isomorphic as (R,S)-groups. Therefore any permutation of G can be extended to an (R,S)-automorphism of F. Now S is a semigroup and G is a group of endomorphisms of F. The d.g. near-ring generated by $SG = \{sg : s \in S, g \in G\}$ in $E(F)$ is denoted by $(R(G),SG)$ and is called the group d.g. near-ring of (R,S) on G.

Theorem 6.1.1. If $r \in (R,S)$, $g \in G$, then $rg = gr$ in $(R(G),SG)$.

Theorem 6.1.2. F is the free $(R(G),SG)$-group on the set X.

Theorem 6.1.3. $(R(G),+)$ considered as an (R,S)-group, is an orthogonal sum of a set $\{ (R_g, +) : g \in G \}$ of (R,S)-groups, where $(R_g, +)$ is isomorphic to $(R, +)$ for each $g \in G$.

6.2. A group d.g. near-ring for a d.g. near-ring

Let (R,S) be a d.g. near-ring and let G be a multiplicative group. Then we have the upper faithful d.g. near-ring $(\overline{R},S)$ together with a d.g. near-ring epimorphism $\theta : (\overline{R},S) \longrightarrow (R,S)$. For each $g \in G$, we have groups $\overline{R}_g$, R_g which are isomorphic copies of $(\overline{R}, +)$ and $(R, +)$ respectively. Then R_g is isomorphic to $\overline{R}_g / I_g$ under the induced homomorphism $\theta_g : \overline{R}_g \longrightarrow R_g$, obtained from θ.

Let $\underset{g \in G}{*} \overline{R}_g$ and $\underset{g \in G}{*} R_g$ be the free products of the families $\{ \overline{R}_g : g \in G \}$ and $\{ R_g : g \in G \}$, respectively, of groups. Then there exists a unique epimorphism $\theta^* : \underset{g \in G}{*} \overline{R}_g \longrightarrow \underset{g \in G}{*} R_g$ such that $\theta^* / \overline{R}_g = \theta_g$, for each $g \in G$. Let K be the normal closure of $\underset{g \in G}{*} I_g$ in $\underset{g \in G}{*} \overline{R}_g$. It is easy to see that K is contained in $\operatorname{Ker} \theta^*$.

From now onwards, we write $* \overline{R}_g$ for $\underset{g \in G}{*} \overline{R}_g$ and similar notation for the other free products, since these are convenient.

Lemma 6.2.1. K is the kernel of θ^*.

Proof. Let $\pi : * \overline{R}_g \longrightarrow (* \overline{R}_g) / \operatorname{Ker} \theta^*$ and $\pi' : * \overline{R}_g \longrightarrow (* \overline{R}_g) / K$ be the natural homomorphisms. Since $I_g \subsetneq K$ for each $g \in G$, π' factors through $*(\overline{R}_g / I_g)$ uniquely, as can be seen from the expression:

$$\pi' = (\theta^* a) h \quad , \qquad (1)$$

where a is the isomorphism from $* R_g$ onto $* (\overline{R}_g / I_g)$ and $h : * (\overline{R}_g / I_g) \longrightarrow (* \overline{R}_g) / K$ is an epimorphism.

Since $* R_g \cong (* \overline{R}_g) / \mathrm{Ker}\ \theta^*$, denoting this epimorphism by b we get a unique homomorphism $\kappa : (* \overline{R}_g) / \mathrm{Ker}\ \theta^* \longrightarrow (* \overline{R}_g) / K$ such that

$$b \kappa = a h \, . \qquad (2)$$

Since $K \subseteq \mathrm{Ker}\ \theta^*$, there exists a unique homomorphism $\kappa' : (* \overline{R}_g) / K \longrightarrow (* \overline{R}_g) / \mathrm{Ker}\ \theta^*$, such that

$$\pi' \kappa' = \pi \, . \qquad (3)$$

It is clear that $\pi = \theta^* b \, .$ (4)

Thus we get a commutative diagram:

$$\begin{tikzcd}
{* \overline{R}_g} \arrow[r, "\pi"] \arrow[d, "\theta^*"'] \arrow[dr, "\pi'" near end] & {(* \overline{R}_g) / \mathrm{Ker}\ \theta^*} \arrow[d, dashed, shift right, "\kappa"'] \\
{* R_g} \arrow[ur, "b" near start] \arrow[r, dashed, "a h"'] & {(* \overline{R}_g) / K} \arrow[u, dashed, shift right, "\kappa'"']
\end{tikzcd}$$

Now from (1), (2), (3) and (4) we get

$$\pi \kappa \kappa' = \theta^* b \kappa \kappa' = \theta^* (a h) \kappa' = \pi' \kappa' = \pi , \qquad (5)$$

$$\pi' \kappa' \kappa = \pi \kappa = \theta^* b \kappa = \theta^* (a h) = \pi' . \qquad (6)$$

Since π, π' are epimorphisms, (5) and (6) respectively imply that

$$\kappa\,\kappa' = I_{(*\overline{R}_g)} / \operatorname{Ker}\theta^* , \qquad (7)$$

$$\kappa'\,\kappa = I_{(*\overline{R}_g)} / K . \qquad (8)$$

Hence κ is an isomorphism and we get $K = \operatorname{Ker}\theta^*$.

Let $\overline{F} = Fr(Y,\overline{R},S)$, where $Y = X \times G$ and X is any set. Then, as in § 6.1. , we can define S as a semigroup of endomorphisms, and G as a group of $(\overline{R},S)$-automorphisms of $\overline{F}$ and then construct the group d.g. near-ring $(\overline{R}(G),SG)$. Clearly $(\overline{R}(G),+)$ is a homomorphic image of $*\overline{R}_g$ under the homomorphism

$$\phi : r_{1g_1} + \cdots + r_{ng_n} \longmapsto r_1 g_1 + \cdots + r_n g_n$$

$r_i \in \overline{R}$, $g_i \in G$.

Let J be the ideal of $(\overline{R}(G),SG)$ generated by

$$IG = \{ Ig : g \in G \} .$$

Then by 1.5.10 , J is the normal subgroup of $(\overline{R}(G),+)$ generated by the set

$$(\overline{R}(G))\, IG\, (SG) = \{ \gamma ag ,\ \gamma(ag)(sh) ,\ (ag)(sh) ,\ ag :$$
$$\gamma \in \overline{R}(G) ,\ a \in I ,\ g,h \in G ,\ s \in S \} .$$

Since I is an ideal of $(\overline{R},S)$ and $rg = gr$ in $(\overline{R}(G),SG)$ for all $r \in \overline{R}$, $g \in G$, we get $agsh = a'g' \in IG$. Therefore J is the normal subgroup of $(\overline{R}(G),+)$, generated by the set

$$\{ \gamma ag ,\ ag : \gamma \in \overline{R}(G) ,\ a \in I ,\ g \in G \} .$$

Now we get a quotient d.g. near-ring $(\overline{R}(G),SG)/J$ with the natural homomorphism $\pi : (\overline{R}(G), SG) \longrightarrow (\overline{R}(G), SG)/J$.

Definition 6.2.2. We call $(\overline{R}(G), SG)/J$ the group d.g. near-ring of (R, S) on G .

$(\overline{R}(G), SG)/J$ is independent of X , since $(\overline{R}(G), SG)$ is so.

Theorem 6.2.3. $(\overline{R}(G)/J, +)$ is a homomorphic image of $* R_g$.

Proof. It is easy to see that the homomorphism ϕ defined above, maps $* I_g$ into $Gp\{(\overline{R}(G)\ IG\ (SG)\}$ and hence K into J . Therefore $K \subseteq \operatorname{Ker} \phi\pi$ and so there exists a unique homomorphism $\psi : * R_g \longrightarrow (\overline{R}(G)/J, +)$ such that $\theta^* \psi = \phi\pi$. Clearly ψ is onto. This completes the proof.

Now for (R, S) we have the lower faithful d.g. near-ring $(\underline{R}, \underline{S})$ with a d.g. near-ring epimorphism $\underline{\theta} : (R, S) \longrightarrow (\underline{R}, \underline{S})$. Again as in § 6.1. we can define $\underline{S}$ as a semigroup of endomorphisms and G as a group of $(\underline{R}, \underline{S})$-automorphisms of $\underline{F} = Fr(Y, \underline{R}, \underline{S})$, the free $(\underline{R}, \underline{S})$-group on Y , and then construct the group d.g. near-ring $(\underline{R}(G), \underline{S}G)$. Since $\theta\,\underline{\theta} : (\overline{R}, S) \longrightarrow (\underline{R}, \underline{S})$ is a d.g. near-ring homomorphism $\underline{F}$ is an $(\overline{R}, S)$-group. Therefore there exists a unique $(\overline{R}, S)$-homomorphism $\mu : \overline{F} \longrightarrow \underline{F}$ such that $\mu / Y = I_Y$. Clearly μ is onto and maps $(x, g)r$ onto $(x, g)(r(\theta\,\underline{\theta}))$, for $r \in \overline{R}$.

Let $\alpha : SG \longrightarrow \underline{S}G$ be defined by $s g \longmapsto \underline{s} g = (s \theta\,\underline{\theta}) g$. Then α is a semigroup homomorphism, because from above we get:

$$\{(s_1\ g_1)(s_2\ g_2)\}\ \alpha = \{(s_1\ s_2)(g_1\ g_2)\}\ \alpha = \{(s_1\ s_2)\,\theta\,\underline{\theta}\}(g_1\ g_2)$$

$$= \{(s_1\,\theta\,\underline{\theta})(s_2\,\theta\underline{\theta})\}(g_1\,g_2) = \{(s_1\,\theta\,\underline{\theta})\,g_1\}\{(s_2\,\theta\,\underline{\theta})\,g_2\}$$

$$= \{(s_1\,g_1)\,\alpha\}\{(s_2\,g_2)\,\alpha\}.$$

Now we prove that α extends to a group homomorphism.

Let $\gamma = \epsilon_1 s_1 g_1 + \text{------} + \epsilon_n s_n g_n$ be 0 in $(\overline{R}(G), +)$.

Then $(x, g)\gamma = 0$ in $\overline{F}$ for all $(x, g) \in Y$. Therefore, for each $(x, g) \in Y$, in $\underline{F}$ we have:

$$0 = \{(x, g)\gamma\}\,\mu$$

$$= \{(x, g)(\epsilon_1 s_1 g_1 + \text{------} + \epsilon_n s_n g_n)\}\,\mu$$

$$= \{\epsilon_1 (x, g) s_1 g_1 + \text{-----} + \epsilon_n (x, g) s_n g_n\}\,\mu$$

$$= \{\epsilon_1 (x, g\,g_1)(s_1\,\theta\,\underline{\theta}) + \text{----} + \epsilon_n (x, g\,g_n)(s_n\,\theta\,\underline{\theta})\}$$

$$= \epsilon_1 (x, g)(s_1\,\theta\,\underline{\theta})\,g_1 + \text{----} + \epsilon_n (x, g)(s_n\,\theta\,\underline{\theta})\,g_n$$

$$= (x, g)\{\epsilon_1 (s_1\,\theta\,\underline{\theta})\,g_1 + \text{-----} + \epsilon_n (s_n\,\theta\,\underline{\theta})\,g_n\}$$

$$= (x, g)\{\epsilon_1 (s_1 g_1)\,\alpha + \text{-----} + \epsilon_n (s_n g_n)\,\alpha\}.$$

Therefore $y\,\{\epsilon_1 (s_1 g_1)\alpha + \text{-----} + \epsilon_n (s_n g_n)\,\alpha\} = 0$ in $\underline{F}$ for each $y \in Y$. Hence $\epsilon_1 (s_1 g_1)\,\alpha + \text{-----} + \epsilon_n (s_n g_n)\,\alpha = 0$ in $(\underline{R}(G), \underline{S}G)$. Therefore α extends to a group homomorphism α from $(\overline{R}(G), +)$ to $(\underline{R}(G), +)$. Hence $\alpha : (\overline{R}(G), SG) \longrightarrow (\underline{R}(G), \underline{S}G)$ is a d.g. near-ring homomorphism. It is clear that α is onto.

Now, since $(a\,g)\,\alpha = (a\,\theta\,\underline{\theta})\,g = (0\underline{\theta})\,g = 0\,g = 0$ for each $a \in I$ and each $g \in G$, we get $J \subseteq \operatorname{Ker} \alpha$. So there exists a unique d.g. near-ring homomorphism

$$\beta : (\overline{R}(G), SG) \,/\, J \longrightarrow (\underline{R}(G), \underline{S}G)$$

such that $\pi\beta = \alpha$. Thus we have proved the following result:

Theorem 6.2.4. $(\underline{R}(G), \underline{S}G)$ is a homomorphic image of $(\overline{R}(G), SG) / J$.

Now for each $g \in G$, $(\overline{R}_g + J)/J = \{ r_g + J : r \in \overline{R} \}$ is an additive subgroup of $(\overline{R}(G)/J, +)$ and

$$(\overline{R}(G)/J, +) = \mathrm{Gp}\{ (\overline{R}_g + J)/J : g \in G \}.$$

The groups $\{ (\overline{R}_g + J)/J : g \in G \}$ can be considered as (R,S)-groups in a natural way.

Lemma 6.2.5. For $r \in \overline{R}$, $g \in G$, rg belongs to J if and only if $r \in I$.

Proof. Clearly $r \in I$ implies that $(rg) \in Ig \subseteq J$.

$\sum_{g \in G} \overline{R}_g$, the direct sum of $\{ \overline{R}_g : g \in G \}$, is a homomorphic image of $(\overline{R}(G),+)$. We denote this homomorphism by ξ. Also $\sum_{g \in G} \overline{R}_g$ is isomorphic to $\overline{R}(G)/C$, where C is the cartesian subgroup of $(\overline{R}(G),+)$ (§ 1 [12]). Clearly ξ maps J into $\sum_{g \in G} I_g$ and therefore $J \cap \overline{R}g$ into $I_g \subseteq \sum I_g$, for all $g \in G$. Hence $(rg)\xi = r_g = a_g$ for some $a \in I$, if $rg \in J$. Hence $r = a \in I$. This completes the proof.

Theorem 6.2.6. For each $g \in G$, $(\overline{R}_g + J)/J$ and R_g are isomorphic as (R,S)-groups.

Proof. Define $\eta : (\overline{R}_g + J)/J \longrightarrow R_g$ by $s\,g \longmapsto (s\,\theta)_g$. By the above lemma η is well-defined and injective. Therefore it extends to a group homomorphism $\eta : (\overline{R}_g + J)/J \longrightarrow R_g$, which is clearly onto. Hence we get

$$(\overline{R}_g + J)/J \cong R_g .$$

It is easy to see that η is an (R,S)-homomorphism.

We have proved that $(\overline{R}(G)/J, +)$ is a group generated by a set of (R,S)-groups. It may be interesting to see whether or not $(\overline{R}(G)/J, +)$ is an (R,S)-group itself.

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